Graph Theory I

Note 5

A graph $G = (V,E)$ consists of a set of vertices V and a set of pairs of vertices $(u,v) \in E$ with $u,v \in V$. In a directed graph, an edge $(u,v) \in E$ is directed from u to v. In an undirected graph the pair is unordered. Unless otherwise specified, graphs in this class are undirected and simple (no self-loops or multiple edges).

Degree: An edge (u,v) is incident to u and v. The degree of a vertex v is the number of edges incident to it, denoted $\deg(v)$.

Degree-sum Formula: $\sum_{v \in V} \deg(v) = 2|E|$. The total number of edge vertex incidences is the sum of the degrees by definition of degree, and also twice the number of edges as each edge is incident to 2 vertices.

Path: A sequence of edges with no repeated vertices. Formally, there is a path between u and v when there is a sequence of vertices $u = v_0, \ldots, v_k = v$ where successive vertices are in an edge, i.e., $(v_i, v_{i+1}) \in E$.

Walk: A sequence of edges $\{(v_1,v_2),(v_2,v_3),\ldots(v_{n-1},v_n)\}$ with possibly repeated vertices.

Cycle: A sequence of edges with start = end, and no other repeated vertices.

Tour: A sequence of edges with start = end, and there may be repeated vertices.

Eulerian Tour: A tour that uses every edge in graph exactly once.

Connected: (u,v) are connected in $G = (V,E)$ if there is a path between u and v. A graph is connected if all pairs of vertices are connected.

Bipartite graph: A graph G with two groups of vertices such that all edges are incident to one vertex in each group.

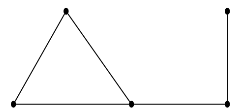
Tree: A graph is a tree iff it satisfies any of the following equivalent conditions:

- connected and acyclic
- connected and has $|V|-1$ edges
- connected, and removing any edge disconnects the graph
- acyclic, and adding any edge creates a cycle

1 Degree Sequences

Note 5

The *degree sequence* of a graph is the sequence of the degrees of the vertices, arranged in descending order, with repetitions as needed. For example, the degree sequence of the following graph is $(3,2,2,2,1)$.



For each of the parts below, determine if there exists a simple undirected graph G (i.e. a graph without self-loops and multiple-edges) having the given degree sequence. Justify your claim.

(a) $(3,3,2,2)$

(b) $(3,2,2,2,2,1,1)$

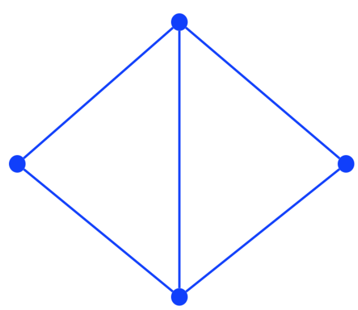
(c) $(6,2,2,2)$

(d) $(4,4,3,2,1)$

Solution:

(a) **Yes**

The following graph has degree sequence $(3,3,2,2)$.



(b) **No**

For any graph G, the number of vertices that have odd degree is even (since the sum of degrees is twice the number of edges). The given degree sequence has 3 odd degree vertices.

(c) **No**

The total number of vertices is 4. Hence there cannot be a vertex with degree 6.

(d) **No**

The total number of vertices is 5. Hence, any degree 4 vertex must have an edge with every other vertex. Since there are two degree 4 vertices, there cannot be a vertex with degree 1.

2 Build-Up Error?

Note 5 What is wrong with the following "proof"? In addition to finding a counterexample, you should explain what is fundamentally wrong with this approach, and why it demonstrates the danger of build-up error.

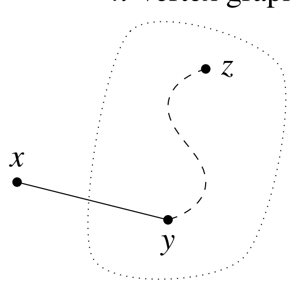
False Claim: If every vertex in an undirected graph with $|V| \geq 2$ has degree at least 1, then it is connected.

Proof? We use induction on the number of vertices $n \geq 2$.

Base case: The only valid graph has two vertices joined by an edge. This graph is connected, so the base case is true.

Inductive hypothesis: Assume the claim is true for some $n \geq 2$.

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x, y\}$ to the path from y to z. This proves the claim for $n+1$. □

Solution:

The mistake is in the argument that *"every $(n+1)$-vertex graph with minimum degree 1 can be obtained from an n-vertex graph with minimum degree 1 by adding 1 more vertex"*. Instead of starting by considering an arbitrary $(n+1)$-vertex graph, this proof only considers an $(n+1)$-vertex graph that you can make by starting with an n-vertex graph with minimum degree 1. As a counterexample, consider a graph on four vertices $V = \{1,2,3,4\}$ with two edges $E = \{\{1,2\},\{3,4\}\}$. Every vertex in this graph has degree 1, but there is no way to build this 4-vertex graph from a 3-vertex graph with minimum degree 1.

More generally, this is an example of *build-up error* in proof by induction. Usually this arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others, such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a *"shrink down, grow back"* process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds.

Let's see what would have happened if we'd tried to prove the claim above by this method. In the inductive step, we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$-vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh! The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck—and properly so, since the claim is false!

3 Coloring Trees

Note 5

(a) Prove that all trees with at least 2 vertices have at least two leaves. Recall that a leaf is defined as a node in a tree with degree exactly 1.

(b) Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* Use induction on the number of vertices.]

Solution:

(a) For an arbitrary tree $T = (V,E)$ where $|V| \geq 2$, suppose L denotes the set of leaves. This means we have $|V| - |L|$ non leaves. A leaf has degree 1 and the other vertices must have degree at least 2. Moreover, we know that an $|V|$-vertex tree must have $|V| - 1$ edges. By the degree-sum formula,

$$2|V| - 2 = \sum_{v \in V} \deg(v) = \sum_{v \in L} \deg(v) + \sum_{v \in V \setminus L} \deg(v) \geq |L| + 2(|V| - |L|) = 2|V| - |L|$$

which implies that $|L| \geq 2$ as desired.

(b) Proof using induction on the number of vertices n.

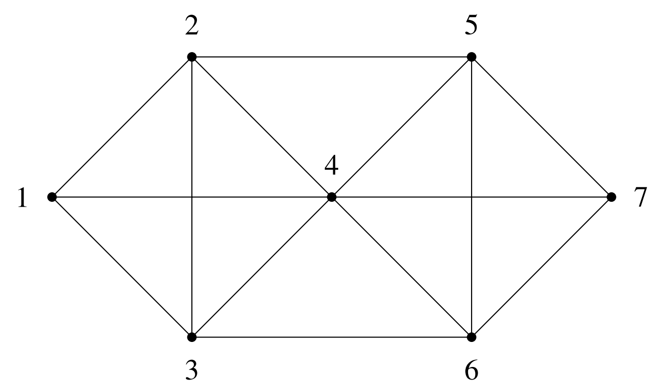
Base case $n = 2$. A tree with two vertices has only one edge and is a bipartite graph by partitioning the two vertices into two separate parts.

Inductive hypothesis. Assume that all trees with k vertices for an arbitrary $k \geq 2$ is bipartite.

Inductive step. Consider a tree $T = (V,E)$ with $k+1$ vertices. We know that every tree must have at least two leaves (previous part), so remove one leaf u and the edge connected to u, say edge e. The resulting graph $T - u$ is a tree with k vertices and is bipartite by the inductive hypothesis. Thus there exists a partitioning of the vertices $V = R \cup L$ such that there does not exist an edge that connects two vertices in L or two vertices in R. Now when we add u back to the graph. If edge e connects u with a vertex in L then let $L' = L$ and $R' = R \cup \{u\}$. On the other hand if edge e connects u with a vertex in R then let $L' = L \cup \{u\}$ and $R' = R$. L' and R' gives us the required partition to show that T is bipartite. This completes the inductive step and hence by induction we get that all trees with at least 2 vertices are bipartite.

4 Eulerian Tour and Eulerian Walk

Note 5



(a) Is there an Eulerian tour in the graph above? If no, give justification. If yes, provide an example.

(b) Is there an Eulerian walk in the graph above? An Eulerian walk is a walk that uses each edge exactly once. If no, give justification. If yes, provide an example.

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

Solution:

(a) No. Two vertices have odd degree.

(b) Yes. One of the two vertices with odd degree must be the starting vertex, and the other one must be the ending vertex. For example: $1 \to 2 \to 3 \to 4 \to 1 \to 3 \to 6 \to 4 \to 5 \to 2 \to 4 \to 7 \to 5 \to 6 \to 7$ will be an Eulerian walk (the numbers are the vertices visited in order). Note that there are 14 edges in the graph.

(c) This solution is long and in depth. Please read slowly, and don't worry if it takes multiple read-throughs since this is dense mathematical text.

An undirected graph has an Eulerian walk if and only if it is connected (except for isolated vertices) and has at most two odd degree vertices. Note that there is no graph with only one odd degree vertex (you can prove this using the degree-sum formula). An Eulerian tour is also an Eulerian walk which starts and ends at the same vertex. We have already seen in the lectures, that an undirected graph G has an Eulerian tour if and only if G is connected (except for isolated vertices) and all its vertices have even degree. We will now prove that a graph

G has an Eulerian walk with distinct starting and ending vertex, if and only if it is connected (except for isolated vertices) and has exactly two odd degree vertices.

Justifications: *Only if.* Suppose there exists an Eulerian walk, say starting at u and ending at v (note that u and v are distinct). Then all the vertices that lie on this walk are connected to each other and all the vertices that do not lie on this walk (if any) must be isolated. Thus the graph is connected (except for isolated vertices). Moreover, every intermediate visit to a vertex in this walk is being paired with two edges, and therefore, except for u and v, all other vertices must be of even degree.

If. First, note that for a connected graph with no odd degree vertices, we have shown in the lectures that there is an Eulerian tour, which implies an Eulerian walk. Thus, let us consider the case of two odd degree vertices.

Solution 1: Take the two odd degree vertices u and v, and add a vertex w with two edges (u,w) and (w,v). The resulting graph G' has only vertices of even degree (we added one to the degree of u and v and introduced a vertex of degree 2) and is still connected. So, we can find an Eulerian tour on G'. Now, delete the component of the tour that uses edges (u,w) and (w,v). The part of the tour that is left is now an Eulerian walk from u to v on the original graph, since it traverses every edge on the original graph.

Solution 2: Alternatively, we can construct an algorithm quite similar to the FindTour algorithm with splicing described in the graphs note.

Suppose G is connected (except for isolated vertices) and has exactly two odd degree vertices, say u and v. First remove the isolated vertices if any. Since u and v belong to a connected component, one can find a path from u to v. Consider the graph obtained by removing the edges of the path from the graph. In the resulting graph, all the vertices have even degree. Hence, for each connected component of the residual graph, we find an Eulerian tour. (Note that the graph obtained by removing the edges of the path can be disconnected.) Observe that an Eulerian walk is simply an edge-disjoint walk that covers all the edges. What we just did is decomposing all the edges into a path from u to v and a bunch of edge-disjoint Eulerian tours. A path is clearly an edge-disjoint walk. Then, given an edge-disjoint walk and an edge-disjoint tour such that they share at least one common vertex, one can combine them into an edge-disjoint walk simply by augmenting the walk with the tour at the common vertex. Therefore we can combine all the edge-disjoint Eulerian tours into the path from u to v to make up an Eulerian walk from u to v.