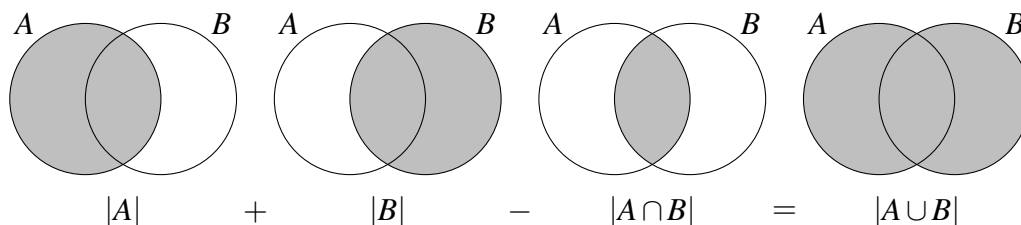


## Counting Intro II

Note 12

**Inclusion-exclusion:** With two sets,



With more sets,

$$\begin{aligned}
 |A_1 \cup A_2 \cup \dots \cup A_n| &= |A_1| + |A_2| + \dots + |A_n| \\
 &\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - \dots - |A_i \cap A_j| - \dots - |A_{n-1} \cap A_n| \\
 &\quad + |A_1 \cap A_2 \cap A_3| + \dots + |A_i \cap A_j \cap A_k| + \dots + |A_{n-2} \cap A_{n-1} \cap A_n| \\
 &\quad \dots \\
 \left| \bigcup_{i=1}^n A_i \right| &= \sum_{k=1}^n (-1)^{k-1} \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S|=k}} \left| \bigcap_{i \in S} A_i \right|
 \end{aligned}$$

That is, for each size  $k$ , iterate through all ways of picking  $k$  sets from  $\{A_1, \dots, A_n\}$ , and alternate between adding and subtracting the sizes of their intersection.

**Combinatorial proofs:** A technique for proving combinatorial identities. There should be very little math involved (usually none): use two different ways of counting the same scenario. One way should correspond to the left-hand side of the equality, and the other way should correspond to the right-hand side of the equality. The fact that we're counting the same scenario means that the two sides are equal.

## 1 Inclusion and Exclusion

Note 12

What is the total number of natural numbers strictly less than 100 that are also coprime to 100?

**Solution:** It is sufficient to count the opposite: what is the total number of natural numbers strictly less than 100 and *not* coprime to 100?

If a number is not coprime to 100, this means that the number is either a multiple of 2 or a multiple of 5. In this case, we have:

- 50 multiples of 2
- 20 multiples of 5
- 10 multiples of both 2 and 5

By inclusion-exclusion, the total number of natural numbers not coprime to 100 is  $50 + 20 - 10 = 60$ , and there are 100 natural numbers strictly less than 100.

As such, in total there are  $100 - 60 = 40$  different natural numbers strictly less than 100 that are coprime to 100.

## 2 CS70: The Musical

### Note 12

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

- (a) First, Edward would like to select directors for his musical. He has received applications from  $2n$  directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

- (b) Edward would now like to select a crew out of  $n$  people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

- (c) There are  $n$  actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}$$

- (d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^n \binom{n}{k} \binom{k}{j} = 2^{n-j} \binom{n}{j}.$$

### Solution:

- (a) Say that we would like to select 2 directors.

**LHS:** This is the number of ways to choose 2 directors out of the  $2n$  candidates.

**RHS:** Split the  $2n$  directors into two groups of  $n$ ; one group consisting of experienced directors, or inexperienced directors (you can split arbitrarily). Then, we consider three cases: either we choose:

1. Both directors from the group of experienced directors,
2. Both directors from the group of inexperienced directors, or
3. One experienced director and one inexperienced director.

The number of ways we can do each of these things is  $\binom{n}{2}$ ,  $\binom{n}{2}$ , and  $n^2$ , respectively. Since these cases are mutually exclusive and cover all possibilities, it must also count the total number of ways to choose 2 directors out of the  $2n$  candidates. This completes the proof.

- (b) Say that we would like to select  $k$  crew members.

**LHS:** This is simply the number of ways to choose  $k$  crew members out of  $n$  candidates.

**RHS:** We select the  $k$  crew members in a different way. First, Edward looks at the first candidate he sees and decides whether or not he would like to choose the candidate. If he selects the first candidate, then Edward needs to choose  $k - 1$  more crew members from the remaining  $n - 1$  candidates. Otherwise, he needs to select all  $k$  crew members from the remaining  $n - 1$  candidates.

We are not double counting here - since in the first case, Edward takes the first candidate he encounters, and in the other case, we do not.

- (c) In this part, Edward selects a subset of the  $n$  actors to be in his musical. Additionally, assume that he must select one individual as the lead for his musical.

**LHS:** Edward casts  $k$  actors in his musical, and then selects one lead among them (note that  $k = \binom{k}{1}$ ). The summation is over all possible sizes for the cast - thus, the expression accounts for all subsets of the  $n$  actors.

**RHS:** From the  $n$  people, Edward selects one lead for his musical. Then, for the remaining  $n - 1$  actors, he decides whether or not he would like to include them in the cast.  $2^{n-1}$  represents the amount of (possibly empty) subsets of the remaining actors. (*Note that for each actor, Edward has 2 choices: to include them, or to exclude them.*)

- (d) In this part, Edward selects a subset of the  $n$  actors to be in the musical; additionally he must select  $j$  lead actors (instead of only 1 in the previous part).

**LHS:** Edward casts  $k \geq j$  actors in his musical, then selects the  $j$  leads among them. Again, the summation is over all possible sizes for the cast (note that any cast that has  $< j$  members is invalid, since Edward would be unable to select  $j$  lead actors) - thus, the expression accounts for all valid subsets of the  $n$  actors.

**RHS:** From the  $n$  people, Edward selects  $j$  leads for his musical. Then, for the remaining  $n - j$  actors, he decides whether or not he would like to include them in the cast. Then, for the remaining  $n - j$  actors, he decides whether or not he would like to include them in the cast.  $2^{n-j}$  represents the amount of ways that Edward can do this.