CS 70 Discrete Mathematics and Probability Theory
Spring 2026 Sinclair, Song

DIS 10A

Combinations of Events Intro

Note 14

Independence: Two events are independent if the following (equivalent) conditions are satisfied. The second definition is probably more intuitive - B happening does not affect the probability of A happening.

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$$
$$\mathbb{P}[A \mid B] = \mathbb{P}[A]$$

Product rule: We can find the probability of an intersection of events by enforcing an "ordering" of these events. Here, each successive conditional probability in the product finds the probability of the next event, *conditioned* on all prior events occurring:

$$\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n] = \mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1]\,\mathbb{P}[A_3 \mid A_1 \cap A_2] \cdots \mathbb{P}[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}].$$

Note that this is a generalization of the definition of conditional probability:

$$\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1].$$

Union Bound: The probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs is at most the sum of the probabilities of the individual events:

$$\mathbb{P}[A_1 \cup A_2 \cup \cdots \cup A_n] \le \mathbb{P}[A_1] + \mathbb{P}[A_2] + \cdots + \mathbb{P}[A_n]$$
$$\mathbb{P}[\bigcup_{i=1}^{n} A_i] \le \sum_{i=1}^{n} \mathbb{P}[A_i]$$

with equality when the A_i's are disjoint.

1 Symmetry

Note 10
Note 13
Note 14

In this problem, we will walk you through the idea of *symmetry* and its formal justification. Consider an experiment where you have a bag with m red marbles and $n-m$ blue marbles. You draw marbles from the bag, one at a time without replacement until the bag is empty.

(a) Define the sample space Ω. (No need to write out every element, a brief description is fine). Is this a uniform probability space?

(b) What is the probability that the first marble you draw is red?

(c) Suppose you've drawn all but the final marble, setting each marble aside as you draw it *without looking at it*. We want to find the probability that the final marble left in the bag will be red.

Let A be the event containing outcomes where the first marble is red, and let B be the event containing outcomes where the final marble is red. Describe, in English, a bijective function $f : A \to B$ mapping outcomes in A to outcomes in B, and explain why it is a bijection. Note that there can be multiple valid bijections. A bijection is a one-to-one mapping.

(d) Use the previous parts to find the probability that the final marble will be red.

(e) You repeat the experiment. Find the probability that the last two marbles you draw will be red.

(f) You repeat the experiment again, but this time you see that the first marble you draw is red. Find the probability that the second-to-last marble you draw will also be red.

Solution:

(a) The sample space is the set of all length n sequences with m reds and $n-m$ blues. There are a total of $\binom{n}{m}$ outcomes in the sample space—each length n sequence is identified by choosing the m positions to be red, and the remaining $n-m$ positions must be blue.

The probability of one of these outcomes (m reds followed by $n-m$ blues) is

$$\overbrace{\frac{m}{n}}^{\text{R}} \times \overbrace{\frac{m-1}{n-1}}^{\text{R}} \times \overbrace{\frac{m-2}{n-2}}^{\text{R}} \times \cdots \times \overbrace{\frac{1}{n-(m-1)}}^{\text{R}} \times \overbrace{\frac{n-m}{n-m}}^{\text{B}} \times \overbrace{\frac{n-m-1}{n-m-1}}^{\text{B}} \times \cdots \times \overbrace{\frac{1}{1}}^{\text{B}}$$

where the denominator is the same for any outcome (denotes how many marbles you can pick from each time), and the numerator terms are also the same, just rearranged. This is because the numerators represent how many red (or blue) marbles are available when we pick that red (or blue) marble. For instance, for your first red marble picked, you have m red marbles available, and for your next red marble you have $m-1$ red marbles available, etc.

Another example outcome, with reds and blues intermingled: if we had red, then blue, then all the other reds followed by all the other blues:

$$\overbrace{\frac{m}{n}}^{\text{R}} \times \overbrace{\frac{n-m}{n-1}}^{\text{B}} \times \overbrace{\frac{m-1}{n-2}}^{\text{R}} \times \cdots \times \overbrace{\frac{2}{n-(m-1)}}^{\text{R}} \times \overbrace{\frac{1}{n-m}}^{\text{R}} \times \overbrace{\frac{n-m-1}{n-m-1}}^{\text{B}} \times \cdots \times \overbrace{\frac{1}{1}}^{\text{B}}$$

We can simplify the probability to equal

$$\begin{aligned} \frac{m!(n-m)!}{n!} &= \frac{1}{\frac{n!}{m!(n-m)!}} \\ &= \frac{1}{\binom{n}{m}} \end{aligned}$$

where the numerator must have factors of $m, m-1, \dots 1$ (for the red marbles) which multiply to be $m!$, and similarly for the blue marbles to be $(n-m)!$.

Each outcome has the same probability $\frac{1}{\binom{n}{m}}$, and there are $\binom{n}{m}$ outcomes, so it is a uniform probability space.

(b) Of the n marbles, m are red, giving a probability of $\frac{m}{n}$.

(c) The inputs to f will be sequences of length n with m red draws and $n-m$ blue draws, and the output will be the same sequence except with the first and last draws swapped. This uniquely transforms each sequence of draws with a red marble first into a sequence with a red marble last, and vice versa (f is its own inverse).

(d) Since we have a bijection between the events A and B, they have the same number of outcomes. Additionally, we have a uniform probability space, so $\mathbb{P}[A] = \frac{|A|}{|\Omega|}$ and $\mathbb{P}[B] = \frac{|B|}{|\Omega|}$. But our bijection showed that $|A| = |B|$, so $\mathbb{P}[A] = \mathbb{P}[B]$, which means the probability of drawing a red marble last is the same as the probability of drawing a red marble first, which is $\frac{m}{n}$.

Note: We don't require a uniform probability space in order to apply the idea of symmetry. The mapping f only needs to map outcomes in A to outcomes in B with the same probability. Mathematically, we require that for every $\omega \in A$, we have $\mathbb{P}[\omega] = \mathbb{P}[f(\omega)]$. Then we'd have

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega] = \sum_{\omega \in A} \mathbb{P}[f(\omega)] = \sum_{\omega \in B} \mathbb{P}[\omega] = \mathbb{P}[B]$$

as desired.

(e) By the same logic as before, the probability that the last two marbles are red is the same as the probability that the first two marbles are red, which is $\frac{m}{n} \times \frac{m-1}{n-1} = \frac{m(m-1)}{n(n-1)}$. The explicit bijection swaps the first two marbles with the last two marbles.

(f) After seeing the first marble is red, there are $m-1$ red marbles left and $n-1$ total marbles. By symmetry, the probability that the second-to-last marble will be red is the same as the probability that the second marble will be red, which is $\frac{m-1}{n-1}$.

2 Pairwise Independence

Note 14

Recall that the events A_1, A_2, and A_3 are *pairwise independent* if for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3]$.

Suppose you roll two fair six-sided dice. Let A_1 be the event that the first die lands on 1, let A_2 be the event that the second die lands on 6, and let A_3 be the event that the two dice sum to 7.

(a) Compute $\mathbb{P}[A_1]$, $\mathbb{P}[A_2]$, and $\mathbb{P}[A_3]$.

(b) Are A_1 and A_2 independent?

(c) Are A_2 and A_3 independent?

(d) Are A_1, A_2, and A_3 pairwise independent?

(e) Are A_1, A_2, and A_3 mutually independent?

Solution:

(a) We have that $\mathbb{P}[A_1] = \mathbb{P}[A_2] = \frac{1}{6}$, since we have a $\frac{1}{6}$ probability of getting a particular number on a fair die.

Since there are 36 ways to roll two fair six-sided dice overall, and 6 of these ways to have the two dice sum to 7 (i.e. $\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$), we have $\mathbb{P}[A_3] = \frac{6}{36} = \frac{1}{6}$ as well.

(b) We want to determine whether $\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]$. We already found the probabilities of A_1 and A_2 from part (a), so let's look at $\mathbb{P}[A_1 \cap A_2]$. There's only one possible outcome where the first die is a 1 and the second die is a 6, so this gives a probability of $\mathbb{P}[A_1 \cap A_2] = \frac{1}{36}$.

Since $\mathbb{P}[A_1]\,\mathbb{P}[A_2] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_1 \cap A_2]$, these two events are independent.

(c) We want to determine whether $\mathbb{P}[A_2 \cap A_3] = \mathbb{P}[A_2]\,\mathbb{P}[A_3]$. We already found the probabilities of A_2 and A_3 from part (a), so let's look at $\mathbb{P}[A_2 \cap A_3]$. These two events both occur if the second die lands on a 6, and the two dice sum to 7. There's only one way that this can happen, i.e. the first die must be a 1, so the intersection has probability $\mathbb{P}[A_2 \cap A_3] = \frac{1}{36}$.

Since $\mathbb{P}[A_2]\,\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_2 \cap A_3]$, these two events are independent.

(d) To see whether the three events are pairwise independent, we need to ensure that all pairs of events are independent. We've already checked that A_1 and A_2 are independent, and that A_2 and A_3 are independent, so it suffices to check whether A_1 and A_3 are independent.

Similar to the previous two parts, the intersection $A_1 \cap A_3$ means that the first die must land on a 1, and the two dice sum to 7. There's only one way for this to happen, i.e. the second die must land on a 6, so the probability is $\mathbb{P}[A_1 \cap A_3] = \frac{1}{36}$.

Since $\mathbb{P}[A_1]\,\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_1 \cap A_3]$, these two events are also independent. Since we've now shown that all possible pairs of events are independent, A_1, A_2, and A_3 are indeed pairwise independent.

(e) Mutual independence requires the additional constraint that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3]$. We've found the individual probabilities of these events in part (a), so we only need to compute $\mathbb{P}[A_1 \cap A_2 \cap A_3]$.

Here, we must have that the first die lands on 1, the second die lands on 6, and the sum of the two dice is equal to 7. There's only one way for this to happen, i.e. the first die is a 1 and the second die is a 6, so the probability of the intersection of all three events is $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \frac{1}{36}$.

However, since $\mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216} \neq \frac{1}{36} = \mathbb{P}[A_1 \cap A_2 \cap A_3]$, these three events are not mutually independent.

3 Calculate These... or Else

Note 14

You draw a 5-card hand from a standard 52-card deck, one card at a time, without putting drawn cards back in the deck.

(a) What is the probability of drawing $3\diamondsuit$ for your first card, $4\diamondsuit$ for your second card, $5\diamondsuit$ for your third, $6\diamondsuit$ for your fourth, and $7\diamondsuit$ for your fifth?

(b) What is the probability that your hand contains the cards $\{3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit\}$ (in any order)?

(c) What is the probability that your hand contains $K\diamondsuit$ or $K\clubsuit$?

(d) What is the probability of drawing at least two K's?

(e) Using the union bound, find an upper bound on the probability of drawing at least two cards with the same value.

Solution: In the solutions, we give both an exact probability and its approximate decimal form. You should be able to obtain the exact probabilities by hand—the decimals are only there for context.

(a) We will use the product rule to find the probability.

- Getting a $3\diamondsuit$ on your first card happens with probability $\frac{1}{52}$: the $3\diamondsuit$ is 1 of 52 cards you can draw first.
- Given that you got a $3\diamondsuit$ on your first card, there remain 51 cards, of which 1 is a $4\diamondsuit$. So the probability of getting $4\diamondsuit$ on your second draw is $\frac{1}{51}$.
- Given that your first card is a $3\diamondsuit$ and your second card is a $4\diamondsuit$, there remain 50 cards, of which 1 is a $5\diamondsuit$. The probability of getting a $5\diamondsuit$ on your third draw is $\frac{1}{50}$.
- Analogously, given your first three cards are as above, the probability of your fourth card being $6\diamondsuit$ is $\frac{1}{49}$.
- Given your first four cards are as above, the probability of your fifth card being $7\diamondsuit$ is $\frac{1}{48}$.

Multiplying these probabilities gives us the of getting $(3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit)$ in this order:

$$\begin{aligned}\mathbb{P}[(3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit)] &= \frac{1}{52} \times \frac{1}{51} \times \frac{1}{50} \times \frac{1}{49} \times \frac{1}{48} \\ &= \frac{1}{52 \times 51 \times 50 \times 49 \times 48} \\ &\approx 3.21 \times 10^{-9}\end{aligned}$$

Note that this is the same probability for any ordered sequence of 5 cards from a 52-card deck. If we take the sample space Ω to be the set of all ordered sequences of 5 cards from a 52-card deck, then Ω has size $52 \times 51 \times 50 \times 49 \times 48$. A probability space using Ω and the probabilities above is therefore a uniform probability space.

(b) Using the uniform probability space from part (a), there are exactly $5!$ outcomes (ordered sequences) that give you a hand of $\{3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit\}$. Applying the formula for probability of events over a uniform probability space,

$$\begin{aligned}\mathbb{P}[\{3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit\}] &= \frac{5!}{52 \times 51 \times 50 \times 49 \times 48} \\ &= \frac{1}{\frac{52\times 51\times 50\times 49\times 48}{5!}} \\ &= \frac{1}{\frac{52!}{5!47!}} \\ &= \frac{1}{\binom{52}{5}} \\ &\approx 3.85 \times 10^{-7}\end{aligned}$$

Note that this is the same probability to draw any (unordered) set of 5 cards from a 52-card deck. Recall that there are $\binom{52}{5}$ ways to draw such a 5-card (unordered) hand, and each has the same probability. So we can define a (different) uniform probability space where outcomes are 5-card (unordered) hands.

(c) *Solution 1*: We will use inclusion-exclusion to find the probability.

First, we'll find the probability that your hand contains $K\diamondsuit$. There are $\binom{51}{4}$ ways to choose the other four cards. Thus, the probability that your hand contains $K\diamondsuit$ is $\frac{\binom{51}{4}}{\binom{52}{5}} \approx 0.0962$.

We can do a similar calculation to find the probability that your hand contains $K\clubsuit$, which is the same.

Now, we'll find the probability that your hand contains both $K\diamondsuit$ and $K\clubsuit$. There are $\binom{50}{3}$ ways to choose the other three cards, so the probability that your hand contains both $K\diamondsuit$ and $K\clubsuit$ is $\frac{\binom{50}{3}}{\binom{52}{5}} \approx 0.00754$.

Putting this all together:

$$\begin{aligned}\mathbb{P}[K\diamondsuit \cup K\clubsuit] &= \mathbb{P}[K\diamondsuit] + \mathbb{P}[K\clubsuit] - \mathbb{P}[K\diamondsuit \cap K\clubsuit] \\ &= \frac{\binom{51}{4}}{\binom{52}{5}} + \frac{\binom{51}{4}}{\binom{52}{5}} - \frac{\binom{50}{3}}{\binom{52}{5}} \\ &\approx 0.185.\end{aligned}$$

Solution 2: We will use the probability of the complement.

For the event that your hand contains $K\diamondsuit$ or $K\clubsuit$, its complement is the event that your hand does not contain $K\diamondsuit$ or $K\clubsuit$. There are $\binom{50}{5}$ ways to choose five cards from the other 50 (non-$K\diamondsuit$, non-$K\clubsuit$) cards, so the probability of the complement is $\frac{\binom{50}{5}}{\binom{52}{5}}$.

Thus, the probability that your hand contains $K\diamondsuit$ or $K\clubsuit$ is

$$1-\frac{\binom{50}{5}}{\binom{52}{5}} \approx 0.185.$$

(d) *Solution 1*: There are 3 disjoint ways to have at least two K's:

- Having two K's (and three non-K's): $\binom{4}{2}\binom{48}{3}$ ways
- Having three K's (and two non-K's): $\binom{4}{3}\binom{48}{2}$ ways
- Having four K's (and one non-K): $\binom{4}{4}\binom{48}{1}$ ways

The probability of drawing at least two K's is therefore:

$$\frac{\binom{4}{2}\binom{48}{3}+\binom{4}{3}\binom{48}{2}+\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \approx 0.0417.$$

Solution 2: We will use the probability of the complement.

For the event that your hand contains at least two K's, its complement is the event that your hand has either zero or one K (the two options are disjoint).

- Having zero K's (and five non-K's): $\binom{48}{5}$ ways
- Having one K (and four non-K's): $\binom{4}{1}\binom{48}{4}$ ways

The probability of drawing either zero or one K is $\frac{\binom{48}{5}+\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$. So the probability of drawing at least two K's is

$$1-\frac{\binom{48}{5}+\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} \approx 0.0417.$$

(e) The event of drawing at least two cards of the same value is equal to the union of drawing at least two A's, at least two 2's, ..., at least two K's. Let B_i be the event of drawing at least two i's. We want to bound the probability of $\bigcup_{i\in\{A,2,3,\ldots,K\}} B_i$.

The probability of drawing at least two i's is the same for each i, and we obtained this probability for $i=K$ in part (d).

Using the union bound,

$$\begin{aligned}
\mathbb{P}[\bigcup_{i\in\{A,2,3,\ldots,K\}} B_i] &\leq \sum_{i\in\{A,2,3,\ldots,K\}} \mathbb{P}[B_i] \\
&= 13 \cdot \mathbb{P}[B_K] \\
&= 13 \cdot \left(\frac{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \right) \\
&\approx 13 \cdot 0.0417 \\
&= 0.542
\end{aligned}$$