CS 70 Discrete Mathematics and Probability Theory
Spring 2026 Sinclair, Song

DIS 11B

Variance and Covariance Intro

Note 17

The **Law of the Unconscious Statistician** (LOTUS): for RV X and function f on the range of X:

$$\mathbb{E}[f(X)] = \sum_k f(k) \cdot \mathbb{P}[X = k].$$

Variance: denoted by $\text{Var}(X)$; measure of how much X deviates from its mean, i.e. its spread.

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Properties: for random variables X, Y and constant a,

- $\text{Var}(aX) = a^2 \text{Var}(X)$
- $\text{Var}(X + a) = \text{Var}(X)$
- If X, Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

Variance of sum of (not necessarily independent) indicator variables: Let $X_1, \ldots, X_n$ be indicator variables for events $A_1, \ldots, A_n$, respectively (i.e., $X_i = 1$ if event A_i occurs, and 0 otherwise). The variance of the sum $X = X_1 + \cdots + X_n$ can be calculated as:

$$\text{Var}(X) = \mathbb{E}[(X_1 + \cdots + X_n)^2] - \mathbb{E}[X_1 + \cdots + X_n]^2 = \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j] - \left(\sum_{i=1}^{n} \mathbb{E}[X_i] \right)^2$$

Note that the term $\sum_{i \neq j} \mathbb{E}[X_i X_j]$ is equivalent to $2 \sum_{i<j} \mathbb{E}[X_i X_j]$.

Covariance: measure of the relationship between two random variables X, Y:

$$\text{cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

Properties: for random variables X, Y, Z and constant a,

- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{cov}(X,Y)$
- $\text{cov}(X,X) = \text{Var}(X)$
- $\text{cov}(X,Y) = \text{cov}(Y,X)$
- Bilinearity: $\text{cov}(X + Y, Z) = \text{cov}(X,Z) + \text{cov}(Y,Z)$ and $\text{cov}(aX,Y) = a\,\text{cov}(X,Y)$

Conditional Expectation: The expectation of a random variable X conditioned on an event A:

$$\mathbb{E}[X \mid A] = \sum_x x \cdot \mathbb{P}[(X = x) \mid A].$$

Often, we use $Y = y$ as the given event. In this case, $\mathbb{E}[X \mid Y = y]$ is a function of Y: it takes inputs $y \in Y$ and outputs $f(y) = \mathbb{E}[X \mid Y = y]$. So $f(Y) = \mathbb{E}[X \mid Y]$ is itself a random variable.

Law of Total Expectation: for random variables X, Y (where $\mathbb{E}[|X|] < \infty$):

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$$

1 Dice Variance

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(a) Let X be a random variable representing the outcome of one roll of a fair 6-sided die. What is $\mathrm{Var}(X)$?

(b) Let Z be a random variable representing the average of n rolls of a fair 6-sided die. What is $\mathrm{Var}(Z)$?

Solution:

(a) Recall that $\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$. We can compute each of the individual terms using the definition of expectation:

$$\begin{aligned}
\mathbb{E}[X] &= \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2} \\
\mathbb{E}\left[X^2\right] &= \frac{1}{6}\left(1^2+2^2+3^2+4^2+5^2+6^2\right) \\
&= \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}
\end{aligned}$$

Now, we plug back into the variance expression:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\
&= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}
\end{aligned}$$

(b) Because each die roll is independent of the others, we can utilize the fact that for independent random variables X and Y, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$. Let X_i be a random variable representing the outcome of the ith dice roll. We now have:

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \left(\frac{1}{n}\right)^2 \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) \\
&= \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \mathrm{Var}(X_i) && (X_i\text{'s are independent}) \\
&= \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \frac{35}{12} && (\text{from (a)}) \\
&= \left(\frac{1}{n}\right)^2 \cdot n \cdot \frac{35}{12} = \frac{35}{12n}
\end{aligned}$$

2 Indicator Expectation

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Let A_1, A_2 be two events where $\mathbb{P}[A_1] = p_1, \mathbb{P}[A_2] = p_2, \mathbb{P}[A_1 \cap A_2] = p_3$. Let X_1, X_2 be indicator variables for A_1 and A_2, respectively.

(a) What is $\mathbb{E}[X_1^2]$? (*Hint*: think about what values X_1^2 can take on, with what probability.)

(b) What is $\mathbb{E}[X_1 X_2]$?

Solution:

(a) Since X_1 is an indicator variable for the event A_1:

$$X_1 = \begin{cases} 1 & \text{if } A_1 \text{ occurs, with probability } \mathbb{P}[A_1] = p_1 \\ 0 & \text{if } A_1 \text{ does not occur (with probability } \mathbb{P}[\overline{A_1}] = 1 - p_1) \end{cases}$$

Thus,

$$X_1^2 = \begin{cases} 1^2 = 1 & \text{if } A_1 \text{ occurs, with probability } \mathbb{P}[A_1] = p_1 \\ 0^2 = 0 & \text{if } A_1 \text{ does not occur (with probability } \mathbb{P}[\overline{A_1}] = 1 - p_1) \end{cases}$$

So:

$$\begin{aligned} \mathbb{E}[X_1^2] &= 1^2 \times \mathbb{P}[X_1^2 = 1] + 0^2 \times \mathbb{P}[X_1^2 = 0] \\ &= 1 \times p_1 + 0 \times (1 - p_1) \\ &= p_1 \end{aligned}$$

Notice that we just showed that $X_1^2 = X_1$: both random variables take on the same value in exactly the same cases. (This is a stronger condition than only requiring that X_1^2 and X_1 take on the same values with the same probabilities, i.e., are identically distributed.)

(b)

$$X_1 X_2 = \begin{cases} 1 & \text{if both } A_1 \text{ and } A_2 \text{ occur, with probability } \mathbb{P}[A_1 \cap A_2] = p_3 \\ 0 & \text{if not both } A_1 \text{ and } A_2 \text{ occur (with probability } \mathbb{P}[\overline{A_1 \cap A_2}] = 1 - p_3) \end{cases}$$

So:

$$\begin{aligned} \mathbb{E}[X_1 X_2] &= 1 \times \mathbb{P}[X_1 X_2 = 1] + 0 \times \mathbb{P}[X_1 X_2 = 0] \\ &= \mathbb{P}[X_1 = 1 \cap X_2 = 1] \\ &= p_3 \end{aligned}$$

3 Elevator Variance

Note 17

A building has n upper floors numbered $1,2,\dots,n$, plus a ground floor G. At the ground floor, m people get on the elevator together, and each person gets off at one of the n upper floors uniformly at random and independently of everyone else.

(a) Let S be the number of floors the elevator skips. Express S as a sum of indicator random variables and compute $\mathbb{E}[S]$.

(b) Write S^2 in terms of the indicators you defined in part (a) and compute $\mathbb{E}[S^2]$.

(c) Using your answers to the previous parts, compute $\text{Var}(S)$.

Solution:

(a) Express S as the sum of the indicator variables $I_1,\dots,I_n$, where $I_i = 1$ if the elevator skips floor i. That is, no one gets off on floor i. Thus, we have

$$\mathbb{E}[I_i] = \mathbb{P}[I_i = 1] = \left(\frac{n-1}{n}\right)^m,$$

and from linearity of expectation,

$$\mathbb{E}[S] = \sum_{i=1}^{n} \mathbb{E}[I_i] = n\left(\frac{n-1}{n}\right)^m.$$

(b) To find the variance, we cannot simply sum up $\text{Var}(I_i)$ because the indicator variables are not necessarily independent. However, since $\text{Var}(S) = \mathbb{E}[S^2] - \mathbb{E}[S]^2$, the only piece we don't already know is $\mathbb{E}[S^2]$. We can calculate this by expanding S^2 as $(I_1 + \cdots + I_n)^2$:

$$\begin{aligned}
S^2 &= (I_1 + \cdots + I_n)^2 \\
&= \sum_{i,j} I_i I_j \\
&= \sum_{i}^{n} I_i^2 + \sum_{i \neq j} I_i I_j \\
\mathbb{E}[S^2] &= \mathbb{E}[(I_1 + \cdots + I_n)^2] \\
&= \mathbb{E}\left[\sum_{i,j} I_i I_j\right] \\
&= \sum_{i}^{n} \mathbb{E}[I_i^2] + \sum_{i \neq j} \mathbb{E}[I_i I_j]
\end{aligned}$$

The first term is simple to calculate: since I_i is an indicator, $I_i^2 = I_i$, so we have

$$\mathbb{E}[I_i^2] = \mathbb{E}[I_i] = \mathbb{P}[I_i = 1] = \left(\frac{n-1}{n}\right)^m,$$

meaning that

$$\sum_{i=1}^{n} \mathbb{E}\left[I_i^2\right] = n\left(\frac{n-1}{n}\right)^m.$$

From the definition of the variables I_i, we see that $I_iI_j = 1$ when both I_i and I_j are 1, which means no one gets off the elevator on floor i and floor j. This happens with probability

$$\mathbb{P}[I_i = I_j = 1] = \mathbb{P}[I_i = 1 \cap I_j = 1] = \left(\frac{n-2}{n}\right)^m.$$

Thus we now know

$$\sum_{i \neq j} \mathbb{E}[I_iI_j] = n(n-1)\left(\frac{n-2}{n}\right)^m,$$

and hence

$$\mathbb{E}[S^2] = n\left(\frac{n-1}{n}\right)^m + n(n-1)\left(\frac{n-2}{n}\right)^m.$$

(c) We can now compute the variance using our results from the previous parts:

$$\operatorname{Var}(S) = \mathbb{E}[S^2] - \mathbb{E}[S]^2 = n\left(\frac{n-1}{n}\right)^m + n(n-1)\left(\frac{n-2}{n}\right)^m - n^2\left(\frac{n-1}{n}\right)^{2m}.$$

4 Covariance

Note 17

(a) We have a bag of 5 red and 5 blue balls. We take two balls uniformly at random from the bag without replacement. Let X_1 and X_2 be indicator random variables for the events of the first and second ball being red, respectively. What is $\operatorname{cov}(X_1, X_2)$?

(b) Now, we have two bags A and B, with 5 red and 5 blue balls each. Draw a ball uniformly at random from A, record its color, and then place it in B. Then draw a ball uniformly at random from B and record its color. Let X_1 and X_2 be indicator random variables for the events of the first and second draws being red, respectively. What is $\operatorname{cov}(X_1, X_2)$?

Solution:

(a) We can use the formula $\operatorname{cov}(X_1, X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2]$.

$$\begin{aligned}
\mathbb{E}[X_1] &= \frac{5}{10} \times 1 + \frac{5}{10} \times 0 = \frac{1}{2}, \\
\mathbb{E}[X_2] &= \frac{5}{10} \times 1 + \frac{5}{10} \times 0 = \frac{1}{2}, \\
\mathbb{E}[X_1X_2] &= \frac{5}{10} \cdot \frac{4}{9} \times 1 + \left(1 - \frac{5}{10} \cdot \frac{4}{9}\right) \times 0 = \frac{2}{9}.
\end{aligned}$$

Therefore,

$$\operatorname{cov}(X_1, X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2] = \frac{2}{9} - \frac{1}{2} \times \frac{1}{2} = -\frac{1}{36}.$$

(b) Again, we use the formula $\operatorname{cov}(X_1, X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2]$.

$$\begin{aligned}
\mathbb{E}[X_1] &= \frac{5}{10} \times 1 + \frac{5}{10} \times 0 = \frac{1}{2} \\
\mathbb{E}[X_2] &= \left(\frac{5}{10} \times \frac{6}{11} + \frac{5}{10} \times \frac{5}{11}\right) \times 1 + \left(\frac{5}{10} \times \frac{5}{11} + \frac{5}{10} \times \frac{6}{11}\right) \times 0 = \frac{1}{2} \\
\mathbb{E}[X_1X_2] &= \frac{5}{10} \times \frac{6}{11} \times 1 = \frac{30}{110}.
\end{aligned}$$

Therefore,

$$\mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2] = \frac{30}{110} - \frac{1}{4} = \frac{1}{44}.$$

Note that in part (a), if one event happened, the other would be less likely to happen, and thus the covariance was negative. Similarly, in part (b), if one event happened, the other would be more likely to happen, and thus the covariance was positive.

5 Number Game

Note 17

Sinho and Vrettos are playing a game where they each choose an integer uniformly at random from $[0, 100]$, then whoever has the larger number wins (in the event of a tie, they replay). However, Vrettos doesn't like losing, so he's rigged his random number generator such that it instead picks randomly from the integers between Sinho's number and 100. Let S be Sinho's number and V be Vrettos' number.

(a) What is $\mathbb{E}[S]$?

(b) What is $\mathbb{E}[V \mid S = s]$, where s is any constant such that $0 \le s \le 100$?

(c) What is $\mathbb{E}[V]$?

Solution:

(a) S is a (discrete) uniform random variable between 0 and 100, so its expectation is $\frac{0+100}{2} = 50$.

(b) If $S = s$, we know that V will be uniformly distributed between s and 100. Similar to the previous part, this gives us that $\mathbb{E}[V \mid S = s] = \frac{s+100}{2}$.

(c) With the law of total expectation, we have that

$$\begin{aligned}
\mathbb{E}[V] &= \sum_{s=0}^{100} \mathbb{E}[V \mid S = s] \cdot \mathbb{P}[S = s] \\
&= \sum_{s=0}^{100} \frac{s+100}{2} \cdot \frac{1}{101} \\
&= \frac{1}{202}\left(\sum_{s=0}^{100} s + \sum_{s=0}^{100} 100\right)
\end{aligned}$$

The first summation comes out to $\frac{100(100+1)}{2} = 50 \cdot 101$; the second summation is just adding 100 to itself 101 times, so it comes out to $100 \cdot 101$. Plugging these values in, we get $\mathbb{E}[V] = 75$.

Alternate Solution:
Using the previous part and the Law of Total Expectation, we get

$$\begin{aligned}\mathbb{E}[V] = \mathbb{E}[\mathbb{E}[V \mid S]] &= \mathbb{E}\left[\frac{S+100}{2}\right] \\ &= \frac{\mathbb{E}[S]+100}{2} \\ &= \frac{150}{2} = 75.\end{aligned}$$