Discussion 14B

Markov Chains Intro II

Note 21

Recall that a Markov chain is defined with the following: the state space $\mathscr{X}$, the transition matrix P, and the initial distribution π_0. This implicitly defines a sequence of random variables X_n with distribution π_n, which denote the state of the Markov chain at timestep n. This sequence of random variables also obey the Markov property: the transition probabilities only depend on the current state, and not any prior states.

A **stationary distribution** (or the **invariant distribution**) of a Markov chain is a row vector π such that $\pi P = \pi$ (That is, transitioning does not change the distribution of states.). The previous equation is called the balance equation, and along with the normalization equation $\sum_i \pi(i) = 1$, we can solve for π.

Irreducibility: A Markov chain is *irreducible* if one can reach any state from any other state in a finite number of steps. An irreducible Markov chain is guaranteed to have a unique invariant distribution.

Periodicity: In an irreducible Markov chain, we define the *period* of a state i as

$$d(i) = \gcd\{n > 0 \mid P^n(i,i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}.$$

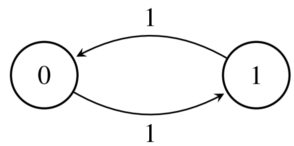
If $d(i) = 1$ for all i, then a Markov chain is *aperiodic*. Otherwise, we say that the Markov chain is *periodic*. One important trait about periods is that they are the same for all states in an irreducible Markov chain. In other words, $d(i) = d(j)$ for all pairs of states i, j.

Fundamental Theorem of Markov Chains: If a Markov chain is irreducible and aperiodic, then for any initial distribution π_0, we have that $\pi_n \to \pi$ as $n \to \infty$, and π is the unique invariant distribution for the Markov chain.

1 Markov Chain Properties

Note 21

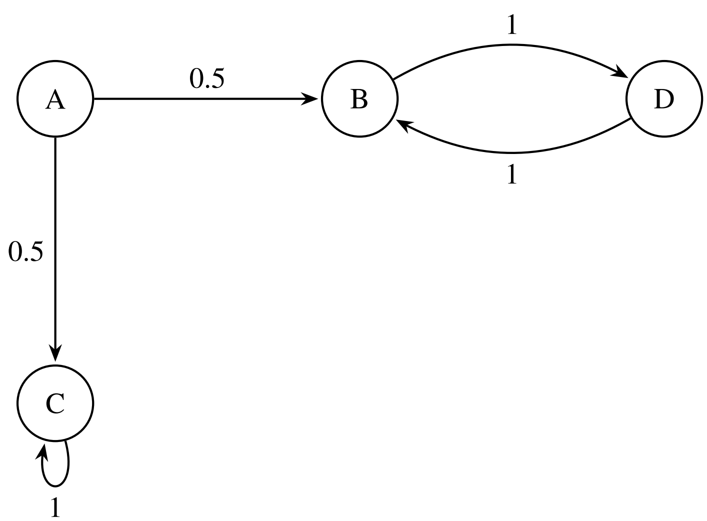
In this question, we will build intuition towards the relationship between irreducibility and the invariant distribution(s), and the relationship between periodicity and convergence to the invariant distribution. Consider the following Markov chain.



(a) What is the period of this Markov chain?

(b) Does this Markov chain have a unique invariant distribution? If so, what is it?

(c) Does the distribution of X_n converge to the invariant distribution as $n \to \infty$ for any initial distribution? If not, give a counterexample and describe how the distribution of X_n behaves as $n \to \infty$.

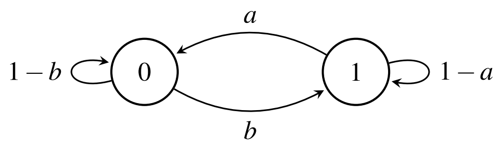


(d) Describe the invariant distribution(s) of this Markov chain. What is the cardinality of the set of invariant distributions?

2 Markov Chain Terminology

Note 21

In this question, we will walk you through terms related to Markov chains. Consider the following Markov chain.



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

(d) Construct a transition probability matrix using the above Markov chain.

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

3 Allen's Umbrella Setup

Note 21

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring exactly one umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is $p > 0$.

(a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix. (*Hint*: You should have 3 states. Keep in mind that our goal is to construct a Markov chain to solve part (c). You may want to try drawing out a 'naive' Markov chain with more states, and then see if you can combine some states together to get a simpler Markov chain.)

(b) Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution.

(c) In the long term, what is the probability that Allen walks through rain with no umbrella?

4 Can it be a Markov Chain?

Note 21

(a) A fly flies in a straight line in unit-length increments. Each second it moves to the left with probability 0.3, right with probability 0.3, and stays put with probability 0.4. There are two spiders at positions 1 and m and if the fly lands in either of those positions it is captured.

Given that the fly starts at state i, where $1 < i < m$, model this process as a Markov Chain. (Don't forget to specify the initial distribution!)

(b) Take the same scenario as in the previous part with $m = 4$. Let $Y_n = 0$ if at time n the fly is in position 1 or 2 and let $Y_n = 1$ if at time n the fly is in position 3 or 4.

Provide the state space for Y_n. Is the process Y_n a Markov chain?