

## 1 Logical Equivalence?

**Note 1** Decide whether each of the following logical equivalences is correct and justify your answer.

(a)  $\forall x (P(x) \wedge Q(x)) \stackrel{?}{\equiv} \forall x P(x) \wedge \forall x Q(x)$

(b)  $\forall x (P(x) \vee Q(x)) \stackrel{?}{\equiv} \forall x P(x) \vee \forall x Q(x)$

(c)  $\exists x (P(x) \vee Q(x)) \stackrel{?}{\equiv} \exists x P(x) \vee \exists x Q(x)$

(d)  $\exists x (P(x) \wedge Q(x)) \stackrel{?}{\equiv} \exists x P(x) \wedge \exists x Q(x)$

### Solution:

(a) **Correct.**

Assume that the left hand side is true. Then we know for an arbitrary  $x$   $P(x) \wedge Q(x)$  is true. This means that both  $\forall x P(x)$  and  $\forall x Q(x)$ . Therefore the right hand side is true. Now for the other direction assume that the right hand side is true. Since for any  $x$   $P(x)$  and for any  $y$   $Q(y)$  holds, then for an arbitrary  $x$  both  $P(x)$  and  $Q(x)$  must be true. Thus the left hand side is true.

(b) **Incorrect.**

Note that there are many possible counterexamples not described here.

Suppose that the universe (i.e. the values that  $x$  can take on) is  $\{1, 2\}$  and that  $P$  and  $Q$  are truth functions defined on this universe. If we set  $P(1)$  to be true,  $Q(1)$  to be false,  $P(2)$  to be false and  $Q(2)$  to be true, the left-hand side will be true, but the right-hand side will be false. Hence, we can find a universe and truth functions  $P$  and  $Q$  for which these two expressions have different values, so they must be different.

Another more concrete example is if  $P(x) = x < 0$  and  $Q(x) = x \geq 0$ , where the universe is the real numbers. For any  $x \in \mathbb{R}$ , exactly one of  $P(x)$  or  $Q(x)$  is true, but it is not the case that  $P(x)$  holds for every  $x$ , and it is also not the case that  $Q(x)$  holds for every  $x$ . Since the LHS and RHS have different values, the two sides are not equivalent.

(c) **Correct**

Assuming that the left hand side is true, we know there exists some  $x$  such that one of  $P(x)$  and  $Q(x)$  is true. Thus  $\exists x P(x)$  or  $\exists x Q(x)$  and the right hand side is true. To prove the other direction, assume the left hand side is false. Then there does not exist an  $x$  for which

$P(x) \vee Q(x)$  is true, which means there is no  $x$  for which  $P(x)$  or  $Q(x)$  is true. Therefore the right hand side is false.

(d) **Incorrect.**

Note, there are many possible counterexamples not described here.

Suppose that the universe (i.e. the values that  $x$  can take on) is the natural numbers  $\mathbb{N}$ , and that  $P$  and  $Q$  are truth functions defined on this universe. Here, suppose we set  $P(1)$  to be true and  $P(x)$  to be false for all other  $x$ , and  $Q(2)$  to be true and  $Q(x)$  to be false for all other  $x$ . (In other words,  $P(x) = (x = 1)$  and  $Q(x) = (x = 2)$ .)

With these definitions, the right hand side would be true, since there exists some value of  $x$  that makes  $P(x)$  true (namely,  $x = 1$ ), and there exists some value of  $x$  that makes  $Q(x)$  true (namely,  $x = 2$ ). However, there would be no value of  $x$  at which both  $P(x)$  and  $Q(x)$  would be simultaneously true, so the left hand side would be false. Hence, we can find a universe and truth functions  $P$  and  $Q$  for which these two expressions have different values, so they must be different.

## 2 Prove

Note 2

Prove each of the following statements

- (a)  $\forall a, b, c \in \mathbb{Z}$ , if  $a|b$  and  $b|c$ , then  $a|c$ .
- (b)  $\forall n \in \mathbb{N}$ ,  $n$  is odd if and only if  $3n + 5$  is even
- (c)  $\forall n \in \mathbb{Z}$ ,  $n^2 + n + 6$  is even.

**Solution:**

- (a) Assume  $a|b$  and  $b|c$ . Then there exist integers  $k_1$  and  $k_2$  such that  $b = k_1a$  and  $c = k_2b$ . Substituting for  $b$  gives  $c = k_2(k_1a) = (k_2k_1)a$ , so  $a|c$ .
- (b) Assume  $n$  is odd. Then  $n = 2k + 1$  for some integer  $k$ . Thus,  $3n + 5 = 3(2k + 1) + 5 = 6k + 3 + 5 = 6k + 8 = 2(3k + 4)$ , which is even. Conversely, if  $3n + 5$  is even, then  $3n + 5 = 2m$  for some integer  $m$ , so  $3n = 2m - 5$ . Since  $2m$  is even,  $2m - 5$  is odd, and thus  $n$  must be odd.
- (c) Assume  $n \in \mathbb{Z}$ . Then  $n^2 + n = n(n + 1)$  is the product of two consecutive integers, which is always even. Thus,  $n^2 + n + 6$  is even.

## 3 Prove or Disprove

Note 2

For each of the following, either prove the statement, or disprove by finding a counterexample.

- (a)  $(\forall n \in \mathbb{N})$  if  $n$  is odd then  $n^2 + 4n$  is odd.
- (b)  $(\forall a, b \in \mathbb{R})$  if  $a + b \leq 15$  then  $a \leq 11$  or  $b \leq 4$ .
- (c)  $(\forall r \in \mathbb{R})$  if  $r^2$  is irrational, then  $r$  is irrational.

- (d)  $(\forall n \in \mathbb{Z}^+) 5n^3 > n!$ . (Note:  $\mathbb{Z}^+$  is the set of positive integers)
- (e) The product of a non-zero rational number and an irrational number is irrational.

**Solution:**

- (a) **Answer:** True.

*Proof.* We will use a direct proof. Assume  $n$  is odd. By the definition of odd numbers,  $n = 2k + 1$  for some natural number  $k$ . This means that we have

$$\begin{aligned} n^2 + 4n &= (2k + 1)^2 + 4(2k + 1) \\ &= 4k^2 + 12k + 5 \\ &= 2(2k^2 + 6k + 2) + 1 \end{aligned}$$

Since  $2k^2 + 6k + 2$  is a natural number, by the definition of odd numbers,  $n^2 + 4n$  is odd.

Alternatively, we could also factor the expression to get  $n(n + 4)$ . Since  $n$  is odd,  $n + 4$  is also odd. The product of 2 odd numbers is also an odd number. Hence  $n^2 + 4n$  is odd.  $\square$

- (b) **Answer:** True.

*Proof.* We will use a proof by contraposition. Suppose that  $a > 11$  and  $b > 4$  (note that this is equivalent to  $\neg(a \leq 11 \vee b \leq 4)$ ). Since  $a > 11$  and  $b > 4$ ,  $a + b > 15$  (note that  $a + b > 15$  is equivalent to  $\neg(a + b \leq 15)$ ). Thus, if  $a + b \leq 15$ , then  $a \leq 11$  or  $b \leq 4$ .  $\square$

- (c) **Answer:** True.

*Proof.* We will use a proof by contraposition. Assume that  $r$  is rational. Since  $r$  is rational, it can be written in the form  $\frac{a}{b}$  where  $a$  and  $b$  are integers with  $b \neq 0$ . Then  $r^2$  can be written as  $\frac{a^2}{b^2}$ . By the definition of rational numbers,  $r^2$  is a rational number, since both  $a^2$  and  $b^2$  are integers, with  $b \neq 0$ . By contraposition, if  $r^2$  is irrational, then  $r$  is irrational.  $\square$

- (d) **Answer:** False.

*Proof.* We will show a counterexample. Let  $n = 7$ . Here,  $5 \cdot 7^3 = 1715$ , but  $7! = 5040$ . Since  $5n^3 < n!$ , the claim is false.

A counterexample that is easier to see without much calculation is for a much larger number like  $n = 100$ ; here,  $100!$  is clearly more than  $5 \cdot 100^3 = 100 \cdot 50 \cdot 25 \cdot 5 \cdot 4 \cdot 2$ , since the latter product contains only a subset of the terms in  $100!$ .  $\square$

- (e) **Answer:** True.

*Proof.* We prove the statement by contradiction. Suppose that  $ab = c$ , where  $a \neq 0$  is rational,  $b$  is irrational, and  $c$  is rational. Since  $a$  and  $b$  are not zero (because 0 is rational),  $c$  is also non-zero. Thus, we can express  $a = \frac{p}{q}$  and  $c = \frac{r}{s}$ , where  $p, q, r$ , and  $s$  are nonzero integers. Then

$$b = \frac{c}{a} = \frac{rq}{ps},$$

which is the ratio of two nonzero integers, giving that  $b$  is rational. This contradicts our initial assumption, so we conclude that the product of a nonzero rational number and an irrational number is irrational.  $\square$

## 4 Induction Starter

Note 3

Consider the inequality  $2^n < n!$  (the right hand side is a factorial, not an exclamation mark).

- (a) Make a conjecture as to which  $n \in \mathbb{N}$  will have the inequality hold.

We will now prove your conjecture using induction.

- (b) What is your base case?  
(c) What is the inductive hypothesis for this proof?  
(d) What do we want to show in the inductive step?  
(e) Conclude the proof with the inductive step.

### Solution:

- (a) Intuitively, we can see that for larger values of  $n$ , as  $n$  increases, the RHS will be multiplied by  $n$  (which continues to grow larger), but the LHS will be multiplied by 2 (which is a constant value). Therefore, we expect that the inequality will hold for sufficiently large  $n$ . We can test out the first few values of  $n$  to see:

- For  $n = 1$ :  $2^1 = 2$  and  $1! = 1$ , so  $2^1 \not< 1!$ .
- For  $n = 2$ :  $2^2 = 4$  and  $2! = 2$ , so  $2^2 \not< 2!$ .
- For  $n = 3$ :  $2^3 = 8$  and  $3! = 6$ , so  $2^3 \not< 3!$ .
- For  $n = 4$ :  $2^4 = 16$  and  $4! = 24$ , so  $2^4 < 4!$ .

Thus, our conjecture is that the inequality  $2^n < n!$  holds for all  $n \geq 4$ .

- (b) The base case is  $n = 4$ . Copying from above, for  $n = 4$ :  $2^4 = 16$  and  $4! = 24$ , so  $2^4 < 4!$ .  
(c) The inductive hypothesis is that the inequality holds for some arbitrary integer  $k \geq 4$ , i.e., we assume that  $2^k < k!$ .  
(d) In the inductive step, we want to show that if the inequality holds for  $k$ , then it also holds for  $k + 1$ . Specifically, we want to prove that  $2^{k+1} < (k + 1)!$ .  
(e) We have:

$$2^{k+1} = 2 \cdot 2^k < 2 \cdot k! \quad (\text{by the inductive hypothesis})$$

Now we need to show that  $2 \cdot k! < (k + 1)!$ . We can rewrite  $(k + 1)!$  as:

$$(k + 1)! = (k + 1) \cdot k!$$

Thus, it suffices to show that  $2 < k + 1$ , which is true for all  $k \geq 4$ .

Therefore, by induction, we have shown that  $2^n < n!$  for all  $n \geq 4$ .

## 5 A Coin Game

Note 3

Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of  $n$  coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be  $3 \cdot 2 = 6$ ). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of  $n$  coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be  $\frac{n(n-1)}{2}$ . (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

### Solution:

We can prove this by strong induction on  $n$ .

**Base Case:** If  $n = 1$ , you start with a stack of one coin, so the game immediately terminates. Your total score is zero—and indeed,  $\frac{n(n-1)}{2} = \frac{1 \cdot 0}{2} = 0$ .

**Inductive Step:** Suppose that if you start with  $i$  coins (for  $i$  between 1 and  $n$  inclusive), your score will be  $\frac{i(i-1)}{2}$  no matter what strategy you employ. Now suppose you start with  $n + 1$  coins. In your first move, you must split your stack into two smaller stacks. Call the sizes of these stacks  $s_1$  and  $s_2$  (so  $s_1 + s_2 = n + 1$  and  $s_1, s_2 \geq 1$ ). Your end score comes from three sources: the points you get from making this first split, the points you get from future splits involving coins from stack 1, and the points you get from future splits involving coins from stack 2. From the rules of the game, we know you get  $s_1 s_2$  points from the first split. From the inductive hypothesis (which we can apply because  $s_1$  and  $s_2$  are between 1 and  $n$ ), we know that the total number of points you get from future splits of stack 1 is  $\frac{s_1(s_1-1)}{2}$  and similarly that the total number of points you get from future splits of stack 2 is  $\frac{s_2(s_2-1)}{2}$ , regardless of what strategy you employ in splitting them. Thus, the total number of points we score is

$$\begin{aligned} s_1 s_2 + \frac{s_1(s_1-1)}{2} + \frac{s_2(s_2-1)}{2} &= \frac{s_1(s_1-1) + 2s_1 s_2 + s_2(s_2-1)}{2} \\ &= \frac{(s_1(s_1-1) + s_1 s_2) + (s_2(s_2-1) + s_1 s_2)}{2} \\ &= \frac{s_1(s_1 + s_2 - 1) + s_2(s_1 + s_2 - 1)}{2} \\ &= \frac{(s_1 + s_2)(s_1 + s_2 - 1)}{2} \end{aligned}$$

Since  $s_1 + s_2 = n + 1$ , this works out to  $\frac{(n+1)(n+1-1)}{2}$ , which is what we wanted to show your total number of points came out to. This completes our proof by induction.

## 6 Proving Inequality

**Note 3** For all positive integers  $n \geq 1$ , prove with induction that

$$\frac{1}{3^1} + \frac{1}{3^2} + \dots + \frac{1}{3^n} < \frac{1}{2}.$$

(Note: while you can use formula for an infinite geometric series to prove this, we require you to use induction. If direct induction seems difficult, consider strengthening the inductive hypothesis. Can you prove an equality statement instead of an inequality?)

**Solution:** Try a few cases and come up with a stronger inductive hypothesis. For example:

- $\frac{1}{3} = \frac{1}{2} - \frac{1}{6}$
- $\frac{1}{3} + \frac{1}{9} = \frac{1}{2} - \frac{1}{18}$
- $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{1}{2} - \frac{1}{54}$

One possible statement is

$$\frac{1}{3^1} + \frac{1}{3^2} + \dots + \frac{1}{3^n} = \frac{1}{2} - \frac{1}{2 \cdot 3^n}$$

- *Base Case:*  $n = 1$ .  $\frac{1}{3} = \frac{1}{2} - \frac{1}{6}$ . True.
- *Inductive Hypothesis:* Assume the statement holds for  $n \geq 1$ .
- *Inductive Step:* Starting from the left hand side,

$$\begin{aligned} \frac{1}{3^1} + \frac{1}{3^2} + \dots + \frac{1}{3^n} + \frac{1}{3^{n+1}} &= \frac{1}{2} - \frac{1}{2 \cdot 3^n} + \frac{1}{3^{n+1}} \\ &= \frac{1}{2} - \frac{3-2}{2 \cdot 3^{n+1}} \\ &= \frac{1}{2} - \frac{1}{2 \cdot 3^{n+1}}. \end{aligned}$$

Therefore,  $\frac{1}{3^1} + \frac{1}{3^2} + \dots + \frac{1}{3^n} = \frac{1}{2} - \frac{1}{2 \cdot 3^n} < \frac{1}{2}$ .

**Alternate Solution:** Normal Induction without strengthening is viable for this problem.

*Base Case:* Suppose  $n = 1$ . We see that  $\frac{1}{3^1} = \frac{1}{3} < \frac{1}{2}$ .

*Inductive Hypothesis:* Suppose the statement is true for some arbitrary  $n = k$ .

*Inductive Step:* Utilizing the hypothesis we get

$$\begin{aligned}\frac{1}{3^1} + \frac{1}{3^2} + \dots + \frac{1}{3^k} + \frac{1}{3^{k+1}} &= \frac{1}{3} + \frac{1}{3} \left( \frac{1}{3^1} + \dots + \frac{1}{3^k} \right) \\ &< \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} \\ &= \frac{1}{2}\end{aligned}$$

which completes the induction.

## 7 Self-Grades

Make sure to review the self grades post on Edstem and submit your selfgrades for the previous homework assignment on Gradescope! This is just a reminder to do so, no need to submit anything for this question.

**Solution:** Submitted the previous homework selfgrades on Gradescope!