CS 70 Discrete Mathematics and Probability Theory
Spring 2026 Sinclair, Song **HW 08**

1 Post-Midterm Check-In Form

Please fill out this required form to let us know how the midterm went for you and what feedback you have for us: https://tinyurl.com/cs70sp26midtermfeedback!

Solution: Filled out the form!

2 Counting Warmup

Note 12

In discussion, we used the combinatorial identity: $\binom{a+b}{a} = \binom{a+b}{b}$. You will prove this important identity in two ways.

(a) Prove this identity algebraically (i.e. using the definition of $\binom{n}{k}$).

(b) Now, prove the identity using a combinatorial proof.

Solution:

(a) From the definition of the binomial coefficient, we know that $\binom{a+b}{a} = \frac{(a+b)!}{(a+b-a)!a!} = \frac{(a+b)!}{b!a!} = \frac{(a+b)!}{a!b!} = \frac{(a+b)!}{(a+b-b)!b!} = \binom{a+b}{b}$.

(b) Suppose we have $a+b$ people to choose a committee of size a from. We can do this in $\binom{a+b}{a}$ ways; the remaining b people are automatically not on a commmittee. Similarly, if we choose the b non-committee people first, the remaining a people are automatically assigned to the committee. This can be done in $\binom{a+b}{b}$ ways.

3 Counting, Counting, and More Counting

Note 12

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) How many 19-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.

(i) How many different 13-card bridge hands are there?

(ii) How many different 13-card bridge hands are there that contain no aces?

(iii) How many different 13-card bridge hands are there that contain all four aces?

(iv) How many different 13-card bridge hands are there that contain exactly 4 spades?

(d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(e) How many 99-bit strings are there that contain more ones than zeros? (Do not leave your answer as a summation. It will simplify nicely, think of how to use symmetry to help you.)

(f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.

(i) How many different anagrams of ALABAMA are there?

(ii) How many different anagrams of MONTANA are there?

(g) How many different anagrams of ABCDEF are there if:

(i) C is the left neighbor of E

(ii) C is on the left of E (and not necessarily E's neighbor)

(h) We have 8 balls, numbered 1 through 8, and 25 bins. How many different ways are there to distribute these 8 balls among the 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(i) How many different ways are there to throw 8 identical balls into 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(j) We throw 8 identical balls into 6 bins. How many different ways are there to distribute these 8 balls among the 6 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 6).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**

Solution:

(a) $\binom{n+k}{k}$

(b) There are 3 options for the first digit. For each of the next digits, they only have 2 options because they cannot be equal to the previous digit. Thus, $3 \cdot 2^{18}$

(c) (i) We have to choose 13 cards out of 52 cards, so this is just $\binom{52}{13}$.

(ii) We now have to choose 13 cards out of 48 non-ace cards. So this is $\binom{48}{13}$.

(iii) We now require the four aces to be present. So we have to choose the remaining 9 cards in our hand from the 48 non-ace cards, and this is $\binom{48}{9}$.

(iv) We need our hand to contain 4 out of the 13 spade cards, and 9 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{4}\binom{39}{9}$ ways to make up the hand.

(d) If we consider the 104! rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number 104! overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52}. Hence, the actual number of rearrangements of 2 identical decks is $\frac{104!}{2^{52}}$.

(e) **Answer 1:** There are $\binom{99}{k}$ 99-bit strings with k ones and $99-k$ zeros. We need $k > 99-k$, i.e. $k \geq 50$. So the total number of such strings is $\sum_{k=50}^{99}\binom{99}{k}$.

This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have

$$\sum_{k=50}^{99}\binom{99}{k} = \sum_{k=50}^{99}\binom{99}{99-k} = \sum_{l=0}^{49}\binom{99}{l}$$

by substituting $l = 99-k$.

Now $\sum_{k=50}^{99}\binom{99}{k} + \sum_{l=0}^{49}\binom{99}{l} = \sum_{m=0}^{99}\binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99}\binom{99}{k} = \frac{1}{2}\cdot 2^{99} = 2^{98}$.

Answer 2 (Symmetry): Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-bit string can have the same number of zeros and ones. Let A be the set of 99-bit strings with more ones than zeros, and B be the set of 99-bit strings with more zeros than ones. Now take any 99-bit string x with more ones than zeros i.e. $x \in A$. If all the bits of x are flipped, then you get a string y with more zeros than ones, and so $y \in B$. This operation of bit flips creates a one-to-one and onto function (called a bijection) between A and B. Hence, it must be that $|A| = |B|$. Every 99-bit string is either in A or in B, and since there are 2^{99} 99-bit strings, we get $|A| = |B| = \frac{1}{2}\cdot 2^{99}$. The answer we sought was $|A| = 2^{98}$.

(f) ALABAMA: The number of ways of rearranging 7 distinct letters and is 7!. In this 7 letter word, the letter A is repeated 4 times while the other letters appear once. Hence, the number 7! overcounts the number of different anagrams by a factor of 4! (which is the number of ways of permuting the 4 A's among themselves). Aka, we only want $1/4!$ out of the total rearrangements. Hence, there are $\frac{7!}{4!}$ anagrams.

MONTANA: In this 7 letter word, the letter A and N are each repeated 2 times while the other letters appear once. Hence, the number 7! overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of 2! for the number of ways of permuting the 2 A's among themselves and another factor of 2! for the number of ways of permuting the 2 N's among themselves). Hence, there are $\frac{7!}{(2!)^2}$ different anagrams.

(g) (i) Suppose we consider CE to be a new letter X; with this replacement, the question is just to count the number of rearrangements of 5 distinct letters, which is 5!.

(ii) Symmetry: Let A be the set of all the rearranging of ABCDEF with C on the left side of E, and B be the set of all the rearranging of ABCDEF with C on the right side of E. $|A \cup B| = 6!, |A \cap B| = 0$. There is a bijection between A and B that can be shown by constructing a operation of exchange the position of C and E. Thus $|A| = |B| = \frac{6!}{2}$.

(h) Each ball has a choice of which bin it should go to. So each ball has 25 choices and the 8 balls can make their choices separately. Hence, there are 25^8 ways.

(i) Since there is no restriction on how many balls a bin needs to have, this is just the problem of throwing k identical balls into n distinguishable bins, which can be done in $\binom{n+k-1}{k}$ ways. Here $k = 8$ and $n = 25$, so there are $\binom{32}{8}$ ways.

(j) **Answer 1:** Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 6 distinguishable bins. There are 2 cases to consider:

Case 1: The 2 balls land in the same bin. This gives 6 ways.

Case 2: The 2 balls land in different bins. This gives $\binom{6}{2}$ ways of choosing 2 out of the 6 bins for the balls to land in. Note that it is *not* 6×5 since the balls are identical and so there is no order on them.

Summing up the number of ways from both cases, we get $6 + \binom{6}{2}$ ways.

Answer 2: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 6 distinguishable bins. From class (see note 10), we already saw that the number of ways to put k identical balls into n distinguishable bins is $\binom{n+k-1}{k}$. Taking $k = 2$ and $n = 6$, we get $\binom{7}{2}$ ways to do this.

EXERCISE: Can you give an expression for the number of ways to put k identical balls into n distinguishable bins such that no bin is empty?

(k) **Answer 1:** Let's number the students from 1 to 20. Student 1 has 19 choices for her partner. Let i be the smallest index among students who have not yet been assigned partners. Then no matter what the value of i is (in particular, i could be 2 or 3), student i has 17 choices for her partner. The next smallest indexed student who doesn't have a partner now has 15 choices for her partner. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \cdots \times 1 = \prod_{i=1}^{10}(2i-1)$.

Answer 2: Arrange the students numbered 1 to 20 in a line. There are 20! such arrangements. We pair up the students at positions $2i-1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the 20! permutations of the students doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of students. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any student pair (x,y), student x could have appeared in position $2i-1$ and student y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the $20!$ permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, $20!$ overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $\frac{20!}{10! \cdot 2^{10}}$ pairings.

Answer 3: In the first step, pick a pair of students from the 20 students. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of students from the remaining 18 students. There are $\binom{18}{2}$ ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of students from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2}\binom{18}{2}\ldots\binom{2}{2}$. However, in any particular pairing of 20 students, this pairing could have been generated in $10!$ ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, ..., tenth step. Hence, we have to divide the above number by $10!$ to get the number of different pairings. Thus there are $\binom{20}{2}\binom{18}{2}\cdots\binom{2}{2}/10!$ different pairings of 20 students.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

4 Proofs of the Combinatorial Variety

Note 12

Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a,b,c) such that $a+b+c=m$.)

Solution:

(a) Suppose we have n people and want to pick some of them to form a special committee. Moreover, suppose we want to pick a leader from among the committee members - how many ways can we do this?

We can do so by first picking the committee members, and then choosing the leader from among the chosen members. We can pick a committee of size k in $\binom{n}{k}$ ways, and once we have picked the committee, we have k choices for which member becomes the leader. In order to account for all possible committee sizes, we need to sum over all valid values of k, hence we get the expression

$$\sum_{k=0}^{n} k \cdot \binom{n}{k},$$

which is exactly the left hand side of the identity we want to prove.

Now, we can also count this set by first picking the leader for the committee, then choosing the rest of committee. We have n choices for the leader, and then among the remaining $n-1$ people, we can pick any subset to form the rest of the committee. Picking a subset of size k can be done in $\binom{n-1}{k}$ ways, hence summing over k, we get the expression

$$n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k},$$

which is exactly the right hand side of the identity we want to prove.

(b) Suppose we have n distinguishable red pencils, n distinguishable blue pencils, and n distinguishable green pencils ($3n$ pencils total), and want to choose m of these pencils to bring to class. How many ways can be do this?

We can do so by just picking the m pencils without considering color, as they are all distinguishable. There are $\binom{3n}{m}$ ways of doing this, which is exactly the right hand side of the identity we want to prove.

We can also count this set by picking some red pencils, the picking some blue pencils, and then finally picking some green pencils. We can pick a red pencils, b blue pencils, and c green pencils (with the tacit assumption that $a+b+c=m$) in $\binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c}$ ways. Finally, in order to account for all possible distributions of pencils, we need to sum over all valid triples (a,b,c), which gives us the expression

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c},$$

which is exactly the left hand side of the identity we want to prove.

5 Five Up

Note 13

Say you toss a coin five times, and record the outcomes. For the three questions below, you can assume that order matters in the outcome, and that the probability of heads is some p in $0 < p < 1$, but *not* that the coin is fair ($p = 0.5$).

(a) What is the size of the sample space, $|\Omega|$?

(b) How many elements of Ω have exactly three heads?

(c) How many elements of Ω have three or more heads?

For the next three questions, you can assume that the coin is fair (i.e. heads comes up with $p = 0.5$, and tails otherwise).

(d) What is the probability that you will observe the sequence HHHTT? What about HHHHT?

(e) What is the probability of observing at least one head?

(f) What is the probability you will observe more heads than tails?

Solution:

(a) Since for each coin toss, we can have either heads or tails, we have 2^5 total possible outcomes.

(b) Since we know that we have exactly 3 heads, what distinguishes the outcomes is at which point these heads occurred. There are 5 possible places for the heads to occur, and we need to choose 3 of them, giving us the following result: $\binom{5}{3}$.

(c) We can use the same approach from part (b), but since we are asking for 3 or more, we need to consider the cases of exactly 4 heads, and exactly 5 heads as well. This gives us the result as: $\binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 16$.

To see why the number is exactly half of the total number of outcomes, denote the set of outcomes that has 3 or more heads as A. If we flip over every coin in each outcome in set A, we get all the outcomes that have 2 or fewer heads. Denote the new set $\overline{A}$. Then we know that A and $\overline{A}$ have the same size and they together cover the whole sample space. Therefore, $|A| = |\overline{A}|$ and $|A| + |\overline{A}| = 2^5$, which gives $|A| = 2^5/2$.

(d) There are $2^5 = 32$ possible coin toss sequences and each of them is equally likely. Hence, each specific sequence will have probability $\frac{1}{32}$. Alternatively, since each coin toss is an independent event, the probability of each of the coin tosses is $\frac{1}{2}$ making the probability of this outcome $\frac{1}{2^5}$. This holds for both cases since both heads and tails have the same probability.

(e) We will use the complementary event, which is the event of getting no heads. The probability of getting no heads is the probability of getting all tails. This event has a probability of $\frac{1}{2^5}$ by a similar argument to the previous part. Since we are asking for the probability of getting at least one heads, our final result is: $1 - \frac{1}{2^5}$.

(f) To have more heads than tails is to claim that we flip at least 3 heads. Since each outcome in this probability space is equally likely, we can divide the number of outcomes where there are 3 or more heads by the total number of outcomes to give us: $\frac{\binom{5}{3}+\binom{5}{4}+\binom{5}{5}}{2^5} = \frac{1}{2}$

Alternatively, we see that for every sequence with more heads than tails we can create a corresponding sequence with more tails than heads by "flipping" the bits. For example, a sequence HTHHT which has more heads than tails corresponds to a flipped sequence THTTH which has more tails than heads. As a result, for every sequence with more heads there's a sequence with more tails. Thus, the probability of having a sequence with more heads is $1/2$.

6 Aces

Note 13

Consider a standard 52-card deck of cards, which has 4 suits (hearts, diamonds, clubs, and spades) with 13 cards in each suit. Each suit has one ace. Hearts and diamonds are red, while clubs and spades are black.

(a) Find the probability of getting an ace or a red card, when drawing a single card.

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

Solution:

(a) Inclusion-Exclusion Principle: $\frac{4}{52}+\frac{26}{52}-\frac{2}{52}=\frac{28}{52}=\frac{7}{13}$.

(b) Inclusion-Exclusion, but we exclude the intersection: $\frac{4}{52}+\frac{13}{52}-2\cdot\frac{1}{52}=\frac{15}{52}$.

(c) Ace of diamonds is fixed, but the other 4 cards in the hand can be any other card: $\frac{\binom{51}{4}}{\binom{52}{5}}$.

(d) Account for the number of ways to draw 2 aces and the number of ways to draw 3 non-aces: $\frac{\binom{4}{2}\cdot\binom{48}{3}}{\binom{52}{5}}$.

(e) Complement to getting no aces: $\mathbb{P}[\text{at least one ace}]=1-\mathbb{P}[\text{zero aces}]=1-\frac{\binom{48}{5}}{\binom{52}{5}}$.

(f) Complement to getting no aces and no hearts:

$\mathbb{P}[\text{at least one ace OR at least one heart}]=1-\mathbb{P}[\text{zero aces AND zero hearts}]=1-\frac{\binom{36}{5}}{\binom{52}{5}}$. This is because $52-13-3=36$, where 13 is the number of hearts and 3 is the number of non-heart aces.

7 Past Probabilified

Note 13

In this question we review some of the past CS70 topics, and look at them probabilistically.
For the following experiments, define an appropriate sample space Ω, and give the probability function $\mathbb{P}[\omega]$ for each $\omega\in\Omega$. Then compute the probabilities of the events E_1 and E_2.

(a) Fix a prime $p > 2$, and uniformly sample twice with replacement from $\{0, \dots, p-1\}$ (assume we have two $\{0, \dots, p-1\}$-sided fair dice and we roll them). Then multiply these two numbers with each other in $\pmod p$ space.

$$E_1 = \text{The resulting product is } 0.$$
$$E_2 = \text{The product is } (p-1)/2.$$

(b) Make a graph on n vertices by sampling uniformly at random from all possible edges, (assume for each edge we flip a coin and if it is head we include the edge in the graph and otherwise we exclude that edge from the graph).

$$E_1 = \text{The graph is complete.}$$
$$E_2 = \text{vertex } v_1 \text{ has degree } d.$$

Solution:

(a) (i) This is essentially the same as throwing two $\{0, \dots, p-1\}$-sided dice, so one appropriate sample space is $\Omega = \{(i, j) \ : \ i, j \in \mathrm{GF}(p)\}$.

(ii) Since there are exactly p^2 such pairs, the probability of sampling each one is $\mathbb{P}[(i, j)] = 1/p^2$.

(iii) Now in order for the product $i \cdot j$ to be zero, at least one of them has to be zero. There are exactly $2p-1$ such pairs, and so $\mathbb{P}[E_1] = \frac{2p-1}{p^2}$.

(iv) For $i \cdot j$ to equal $(p-1)/2$ it doesn't matter what i is as long as $i \neq 0$ and $j \equiv i^{-1}(p-1)/2 \pmod p$. Since p is a prime, $p-1$ numbers $\pmod p$ will have an inverse $\pmod p$. Thus $|E_2| = \left|\left\{(i, j) \ : \ j \equiv i^{-1}(p-1)/2\right\}\right| = p-1$, and whence $\mathbb{P}[E_2] = \frac{p-1}{p^2}$.
Alternative Solution for $\mathbb{P}[E_2]$: The previous reasoning showed that $(p-1)/2$ is in no way special, and the probability that $i \cdot j = (p-1)/2$ is the same as $\mathbb{P}[i \cdot j = k]$ for any $k \in \mathrm{GF}(p)$. But $1 = \sum_{k=0}^{p-1} \mathbb{P}[i \cdot j = k] = \mathbb{P}[i \cdot j = 0] + (p-1)\mathbb{P}[i \cdot j = (p-1)/2] = \frac{2p-1}{p^2} + (p-1)\mathbb{P}[i \cdot j = (p-1)/2]$, and so $\mathbb{P}[E_2] = \left(1 - \frac{2p-1}{p^2}\right)/(p-1) = \frac{p-1}{p^2}$ as desired.

(b) (i) Since any n-vertex graph can be sampled, Ω is the set of all graphs on n vertices.

(ii) As there are $N = 2^{\binom{n}{2}}$ such graphs, the probability of each indivdual one g is $\mathbb{P}[g] = 1/N$ (by the same reasoning that every sequence of fair coin flips is equally likely!).

(iii) There is only one complete graph on n vertices, and so $\mathbb{P}[E_1] = 1/N$.

(iv) For vertex v_1 to have degree d, exactly d of its $n-1$ possible adjacent edges must be present. There are $\binom{n-1}{d}$ choices for such edges, and for any fixed choice, there are $2^{\binom{n}{2}-(n-1)}$ graphs with this choice. So $\mathbb{P}[E_2] = \frac{\binom{n-1}{d} 2^{\binom{n}{2}-(n-1)}}{2^{\binom{n}{2}}} = \binom{n-1}{d}\left(\frac{1}{2}\right)^{n-1}$.