CS 70 Discrete Mathematics and Probability Theory
Spring 2026 Sinclair, Song

HW 11

1 Balls and Bins, All Day Every Day

Note 14
Note 15

Suppose n balls are thrown into n labeled bins one at a time, where n is a positive *even* integer.

(a) What is the probability that exactly k balls land in the first bin, where k is an integer $0 \le k \le n$?

(b) What is the probability p that at least half of the balls land in the first bin? (You may leave your answer as a summation.)

(c) Using the union bound, give a simple upper bound, in terms of p, on the probability that some bin contains at least half of the balls.

(d) What is the probability, in terms of p, that at least half of the balls land in the first bin, or at least half of the balls land in the second bin?

(e) After you throw the balls into the bins, you walk over to the bin which contains the first ball you threw, and you randomly pick a ball from this bin. What is the probability that you pick up the first ball you threw? (Again, leave your answer as a summation.)

Solution:

(a) The probability that a particular ball lands in the first bin is $1/n$. We need exactly k balls to land in the first bin, which occurs with probability $(1/n)^k$, and we need exactly $n-k$ balls to land in a different bin, which occurs with probability $(1-1/n)^{n-k}$, and there are $\binom{n}{k}$ ways to choose which of the k balls land in first bin. Thus, the probability is $\binom{n}{k}(1/n)^k(1-1/n)^{n-k}$.

(b) This is the summation over $k = n/2, \ldots, n$ of the probabilities computed in the first part, i.e., $\sum_{k=n/2}^{n}\binom{n}{k}(1/n)^k(1-1/n)^{n-k}$.

(c) The event that some bin has at least half of the balls is the union of the events A_k, $k = 1, \ldots, n$, where A_k is the event that bin k has at least half of the balls. By the union bound, $\mathbb{P}(\bigcup_{k=1}^{n} A_k) \le \sum_{k=1}^{n} \mathbb{P}(A_k) = np$.

(d) The probability that the first bin has at least half of the balls is p; similarly, the probability that the second bin has at least half of the balls is also p. There is overlap between these two events, however: the first bin has half of the balls and the second bin has the second half of the balls. The probability of this event is $\binom{n}{n/2}n^{-n}$: there are n^n total possible configurations for the n balls to land in the bins, but if we require exactly $n/2$ of the balls to land in the first bin and the remaining balls to land in the second bin, there are $\binom{n}{n/2}$ ways to choose which balls land in the first bin. By the principle of inclusion-exclusion, our desired probability is $p + p - \binom{n}{n/2}n^{-n} = 2p - \binom{n}{n/2}n^{-n}$.

(e) Condition on the number of balls in the bin. First we calculate the probability $\mathbb{P}(A_k)$, where A_k is the event that, in addition to the first ball you threw, an additional $k-1$ of the other $n-1$ balls landed in this bin, which by the reasoning in Part (a) has probability

$$\mathbb{P}(A_k) = \binom{n-1}{k-1}(1/n)^{k-1}(1-1/n)^{n-k} \ .$$

If we let B be the event that we pick up the first ball we threw, then

$$\mathbb{P}(B \mid A_k) = 1/k$$

since we are equally likely to pick any of the k balls in the bin. Thus the overall probability we are looking for is, by an application of the law of total probability,

$$\mathbb{P}(B) = \sum_{k=1}^{n} \mathbb{P}(A_k \cap B) = \sum_{k=1}^{n} \mathbb{P}(A_k)\,\mathbb{P}(B \mid A_k) = \sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1}\left(\frac{1}{n}\right)^{k-1}\left(1-\frac{1}{n}\right)^{n-k}.$$

2 Combined Head Count

Note 16

Suppose you flip a fair coin twice.

(a) What is the sample space Ω generated from these flips?

(b) Define a random variable X to be the number of heads. What is the distribution of X?

(c) Define a random variable Y to be 1 if you get a heads followed by a tails and 0 otherwise. What is the distribution of Y?

(d) Compute the conditional probabilities $\mathbb{P}[Y = i \mid X = j]$ for all combinations of i and j.

(e) Define a third random variable $Z = X + Y$. Use the conditional probabilities you computed in part (d) to find the distribution of Z.

Solution:

(a) $\{(T,T),(H,T),(T,H),(H,H)\}$.

(b)

$$X = \begin{cases} 0 & \text{w.p.} \quad .25 \\ 1 & \text{w.p.} \quad .5 \\ 2 & \text{w.p.} \quad .25 \end{cases}$$

(c)

$$Y = \begin{cases} 0 & \text{w.p.} \quad .75 \\ 1 & \text{w.p.} \quad .25 \end{cases}$$

(d)
- $\mathbb{P}[Y = 0 \mid X = 0]$: Since $X = 0$, we have no heads; therefore, there is no chance that the first coin is heads, so Y must be 0. So $\mathbb{P}[Y = 0 \mid X = 0] = 1$.

    - $\mathbb{P}[Y=1 \mid X=0] = 0$ as $\mathbb{P}[Y=1 \mid X=0] = 1 - \mathbb{P}[Y=0 \mid X=0] = 1-1=0$.
    - $\mathbb{P}[Y=0 \mid X=1]$: If we have one head, then we have one of two outcomes, (H,T) or (T,H), and since this is a fair coin, both outcomes happen with equal probability. Only (T,H) makes $Y=0$; thus $\mathbb{P}[Y=0 \mid X=1] = \frac{1}{2}$.
    - $\mathbb{P}[Y=1 \mid X=1] = 0$ as $\mathbb{P}[Y=1 \mid X=1] = 1 - \mathbb{P}[Y=0 \mid X=1] = 1 - \frac{1}{2} = \frac{1}{2}$.
    - $\mathbb{P}[Y=0 \mid X=2]$: Since $X=0$, we have no tails; therefore, there is no chance that the second coin is tails, so Y must be 0. So $\mathbb{P}[Y=0 \mid X=2] = 1$.
    - $\mathbb{P}[Y=1 \mid X=2] = 0$ as $\mathbb{P}[Y=1 \mid X=2] = 1 - \mathbb{P}[Y=0 \mid X=2] = 1-1=0$.

(e) Let's determine the values Z can take and the corresponding probabilities:

- $Z=0$: $\mathbb{P}(Z=0) = \mathbb{P}(X=0 \cap Y=0) = \mathbb{P}(X=0) \cdot \mathbb{P}(Y=0 \mid X=0) = .25 \cdot 1 = .25$
- $Z=1$:

$$\begin{aligned}\mathbb{P}(Z=1) &= \mathbb{P}(X=0 \cap Y=1) + \mathbb{P}(X=1 \cap Y=0) \\ &= \mathbb{P}(X=0) \cdot \mathbb{P}(Y=1 \mid X=0) + \mathbb{P}(X=1) \cdot \mathbb{P}(Y=0 \mid X=1) \\ &= .25 \cdot 0 + .5 \cdot .5 = .25\end{aligned}$$

- $Z=2$:

$$\begin{aligned}\mathbb{P}(Z=2) &= \mathbb{P}(X=1 \cap Y=1) + \mathbb{P}(X=2 \cap Y=0) \\ &= \mathbb{P}(X=1) \cdot \mathbb{P}(Y=1 \mid X=1) + \mathbb{P}(X=2) \cdot \mathbb{P}(Y=0 \mid X=2) \\ &= .5 \cdot 5 + .25 \cdot 1 = .5\end{aligned}$$

- $Z=3$: $\mathbb{P}(Z=3) = \mathbb{P}(X=2 \cap Y=1) = \mathbb{P}(X=2) \cdot \mathbb{P}(Y=1 \mid X=2) = .25 \cdot 0 = 0$

$$Z = \begin{cases} 0 & \text{w.p.} \quad .25 \\ 1 & \text{w.p.} \quad .25 \\ 2 & \text{w.p.} \quad .5 \end{cases}$$

3 Testing Model Planes

Note 16

Amin is testing model airplanes. He starts with n model planes which each independently have probability p of flying successfully each time they are flown, where $0 < p < 1$. Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Amin flying all remaining model planes and throwing away any that crash.

Let X_i be the random variable representing how many model planes remain after i days. (Note that $X_0 = n$.) Justify your answers for each part.

(a) What is the distribution of X_1? That is, what is $\mathbb{P}[X_1 = k]$?

(b) What is the distribution of X_2? That is, what is $\mathbb{P}[X_2 = k]$? Recognize the distribution of X_2 as one of the famous ones and provide its name and parameters.

(c) Repeat the previous part for X_t for arbitrary $t \geq 1$.

(d) What is the probability that at least one model plane still remains (has not crashed yet) after t days? Do not have any summations in your answer.

(e) Considering only the first day of flights, is the event A_1 that the first and second model planes crash independent from the event B_1 that the second and third model planes crash? Recall that two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$. Prove your answer using this definition.

(f) Considering only the first day of flights, let A_2 be the event that the first model plane crashes *and* exactly two model planes crash in total. Let B_2 be the event that the second plane crashes on the first day. What must n be equal to in terms of p such that A_2 is independent from B_2? Prove your answer using the definition of independence stated in the previous part.

(g) Are the random variables X_i and X_j, where $i < j$, independent? Recall that two random variables X and Y are independent if $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1]\,\mathbb{P}[Y = k_2]$ for all k_1 and k_2. Prove your answer using this definition.

Solution:

(a) Since Amin is performing n trials (flying a plane), each with an independent probability of "success" (not crashing), we have $X_1 \sim \text{Binomial}(n, p)$, or $\mathbb{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$, for $0 \leq k \leq n$.

(b) Each model plane independently has probability p^2 of surviving both days. Whether a model plane survives both days is still independent from whether any other model plane survives both days, so we can say $X_2 \sim \text{Binomial}(n, p^2)$, or $\mathbb{P}[X = k] = \binom{n}{k} p^{2k} (1-p^2)^{n-k}$, for $0 \leq k \leq n$.

(c) By extending the previous part, we see each model plane has probability p^t of surviving t days, so $X_t \sim \text{Binomial}(n, p^t)$, or $\mathbb{P}[X = k] = \binom{n}{k} (p^t)^k (1-p^t)^{n-k}$, for $0 \leq k \leq n$.

(d) We consider the complement, the probability that no model planes remain after t days. By the previous part we know this to be

$$\mathbb{P}[X_t = 0] = \binom{n}{0} (p^t)^0 (1-p^t)^{n-0} = (1-p^t)^n.$$

This means that the probability of at least model plane remaining after t days is $1-(1-p^t)^n$.

(e) No. $\mathbb{P}[A_1 \cap B_1]$ is the probability that the first three model planes crash, which is $(1-p)^3$. But $\mathbb{P}[A_1]\,\mathbb{P}[B_1] = (1-p)^2(1-p)^2 = (1-p)^4$. So $\mathbb{P}[A_1 \cap B_1] \neq \mathbb{P}[A_1]\,\mathbb{P}[B_1]$ and A_1 and B_1 are not independent.

(f) $\mathbb{P}[A_2 \cap B_2]$ is the probability that only the first model plane and second model plane crash, which is $(1-p)^2 p^{n-2}$. $\mathbb{P}[A_2]$ is the probability that the first model plane crashes, and exactly one of the remaining $n-1$ model planes crashes, so

$$\mathbb{P}[A_2] = (1-p)\cdot\binom{n-1}{1}(1-p)p^{n-1-1} = (n-1)(1-p)^2 p^{n-2}.$$

We also have $\mathbb{P}[B_2] = 1-p$, so we want to solve for n in

$$\begin{aligned}
\mathbb{P}[A_2 \cap B_2] &= \mathbb{P}[A_2]\,\mathbb{P}[B_2] \\
(1-p)^2 p^{n-2} &= \underbrace{(n-1)(1-p)^2 p^{n-2}}_{\mathbb{P}[A_2]}\underbrace{(1-p)}_{\mathbb{P}[B_2]} \\
(1-p)^2 p^{n-2} &= (n-1)(1-p)^3 p^{n-2} \\
1 &= (n-1)(1-p) \\
n &= 1 + \frac{1}{1-p}
\end{aligned}$$

(g) No. Let $k_1 = 0$ and $k_2 = 1$. Then, $\mathbb{P}[X_i = k_1 \cap X_j = k_2] = 0$ because you can't have 1 plane at the end of day 2 if there are no planes left at the end of day 1. However, $\mathbb{P}[X_i = k_1] > 0$ and $\mathbb{P}[X_j = k_2] > 0$, so $\mathbb{P}[X_i = k_1]\,\mathbb{P}[X_j = k_2] > 0$. Since $\mathbb{P}[X_i = k_1]\,\mathbb{P}[X_j = k_2] \neq \mathbb{P}[X_i = k_1 \cap X_j = k_2]$, they are not independent.

4 Max/Min Dice Rolls

Note 16

Yining rolls three fair six-sided dice.

(a) Let X denote the maximum of the three values rolled. What is the distribution of X? That is, what is $\mathbb{P}[X = x]$ for each possible value of x? Leave your final answer in terms of x.

Hint: Try to first compute $\mathbb{P}[X \le x]$ for each possible value of x. To check your answer, you can solve this problem using counting and ensure you get the same probabilities.

(b) Let Y denote the minimum of the three values rolled. What is the distribution of Y?

Solution:

(a) Let X denote the maximum of the three values rolled. We are interested in $\mathbb{P}[X = x]$, for $x = 1,2,3,4,5,6$ (the possible values of the maximum of three dice rolls). First, define X_1, X_2, X_3 to be the values rolled by the first, second, and third dice. These random variables are i.i.d. and uniformly distributed between 1 and 6 inclusive.

Following the hint, we first compute $\mathbb{P}[X \le x]$ for $x = 1,2,3,4,5,6$:

$$\begin{aligned}
\mathbb{P}[X \le x] &= \mathbb{P}[(X_1 \le x) \cap (X_2 \le x) \cap (X_3 \le x)] \\
&= \mathbb{P}[X_1 \le x]\,\mathbb{P}[X_2 \le x]\,\mathbb{P}[X_3 \le x] \\
&= \left(\frac{x}{6}\right)\left(\frac{x}{6}\right)\left(\frac{x}{6}\right) \\
&= \left(\frac{x}{6}\right)^3
\end{aligned}$$

where the second equality comes from the fact that for the maximum to be at most x, we need all three dice rolls to be at most x.

Then, observing that $\mathbb{P}[X = x] = \mathbb{P}[X \le x] - \mathbb{P}[X \le x-1]$:

$$\mathbb{P}[X = x] = \left(\frac{x}{6}\right)^3 - \left(\frac{x-1}{6}\right)^3 = \frac{3x^2 - 3x + 1}{216} = \begin{cases} \frac{1}{216}, & x = 1 \\ \frac{7}{216}, & x = 2 \\ \frac{19}{216}, & x = 3 \\ \frac{37}{216}, & x = 4 \\ \frac{61}{216}, & x = 5 \\ \frac{91}{216}, & x = 6 \end{cases}$$

One can confirm that $\sum_{x=1}^{6} \mathbb{P}[X = x] = 1$.

Alternate Solution: We can also find $\mathbb{P}[X = x]$ by casework.

In order for the maximum to be x, there are three cases on how many dice turn out to be x:

- One die is x, the other two die are less than x. We have $\binom{3}{1}$ ways to choose the one x die, which has a $\frac{1}{6}$ probability of rolling x, and there is a $(\frac{x-1}{6})^2$ chance for both the other dice to roll less than x. The probability of this case is thus $\binom{3}{1}(\frac{1}{6})(\frac{x-1}{6})^2$.
- Two dice are x, the other die is less than x. We have $\binom{3}{2}$ ways to choose the two x dice, which have a $(\frac{1}{6})^2$ probability of both of them rolling x, and the other die has a $\frac{x-1}{6}$ probability of rolling less than x. The probability of this case is thus $\binom{3}{2}(\frac{1}{6})^2(\frac{x-1}{6})$.
- All three dice are x. The probability of this case is $(\frac{1}{6})^3$.

Thus,

$$\mathbb{P}[X = x] = \binom{3}{1}(\frac{1}{6})(\frac{x-1}{6})^2 + \binom{3}{2}(\frac{1}{6})^2(\frac{x-1}{6}) + (\frac{1}{6})^3$$

(b) In a similar fashion as part (a), we first compute $\mathbb{P}[Y \ge y]$.

$$\begin{aligned} \mathbb{P}[Y \ge y] &= \mathbb{P}[(X_1 \ge y) \cap (X_2 \ge y) \cap (X_3 \ge y)] \\ &= \mathbb{P}[X_1 \ge y]\,\mathbb{P}[X_2 \ge y]\,\mathbb{P}[X_3 \ge y] \\ &= \left(\frac{6-(y-1)}{6}\right)\left(\frac{6-(y-1)}{6}\right)\left(\frac{6-(y-1)}{6}\right) \\ &= \left(\frac{7-y}{6}\right)^3 \end{aligned}$$

Then, observing that $\mathbb{P}[Y = y] = \mathbb{P}[Y \geq y] - \mathbb{P}[Y \geq y+1]$:

$$\mathbb{P}[Y = y] = \left(\frac{7-y}{6}\right)^3 - \left(\frac{6-y}{6}\right)^3$$

5 How Many Marbles?

Note 16

Leanne has 6 marbles, 2 red, 2 blue, and 2 green. She picks three marbles uniformly at random without replacement. Let X denote the number of blue marbles she draws.

(a) What is $\mathbb{P}[X = 0]$, $\mathbb{P}[X = 1]$, and $\mathbb{P}[X = 2]$?

(b) What do the probabilities you computed in part (a) add up to?

(c) Compute $\mathbb{E}[X]$, using the definition of expectation.

(d) Suppose we define indicators X_i, $1 \leq i \leq 3$, where X_i is the indicator variable that equals 1 if the ith marble is a blue marble and 0 otherwise. Compute $\mathbb{E}[X]$ using linearity of expectation.

(e) Are the X_i indicators independent? Does this affect your solution to part (d)?

Solution:

(a) Calculate each case of $X = 0, 1$, and 2:

We must draw three non-blue marbles in a row, so the probability is

$$\mathbb{P}[X = 0] = \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} = \frac{1}{5}.$$

Alternatively, every set of three marbles is equally likely, so we can use counting. There are $\binom{6}{3}$ total sets of three marbles, and $\binom{4}{3}$ sets with only non-blue marbles, which gives us the same result.

$$\mathbb{P}[X = 0] = \frac{\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5}.$$

- We will continue to use counting. The number of sets of three marbles with exactly one blue marble amounts to the number of ways to choose 1 blue marble out of 2, and 2 non-blues out of 4.

$$\mathbb{P}[X = 1] = \frac{\binom{2}{1}\binom{4}{2}}{\binom{6}{3}} = \frac{3}{5}$$

- Choose 2 blue marbles out of 2, and 1 non-blue out of 4.

$$\mathbb{P}[X = 2] = \frac{\binom{2}{2}\binom{4}{1}}{\binom{6}{3}} = \frac{1}{5}.$$

(b) We check:

$$\mathbb{P}[X=0]+\mathbb{P}[X=1]+\mathbb{P}[X=2]=\frac{1+3+1}{5}=1$$

(c) From the definition, $\mathbb{E}[X]=\sum_{k=0}^{2} k\,\mathbb{P}[X=k]$, so

$$\mathbb{E}[X]=0\cdot\frac{1}{5}+1\cdot\frac{3}{5}+2\cdot\frac{1}{5}=1.$$

(d) We know that $\mathbb{E}[X_i]=\mathbb{P}[\text{marble } i \text{ is a blue marble}]+0\cdot\mathbb{P}[\text{marble } i \text{ is not a blue marble}]=\frac{2}{6}=\frac{1}{3}$, so

$$\mathbb{E}[X]=\mathbb{E}[X_1]+\mathbb{E}[X_2]+\mathbb{E}[X_3]=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1.$$

Notice how much faster it was to compute the expectation using indicators!

(e) No, they are not independent. As an example:

$$\mathbb{P}[X_1=1]\,\mathbb{P}[X_2=1]=\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{9}$$

However,

$$\mathbb{P}[X_1=1,X_2=1]=\mathbb{P}[\text{the first and second marbles are both blue}]=\frac{2}{6}\cdot\frac{1}{5}=\frac{1}{15}.$$

Even though the indicators are not independent, this does not change our answer for part (d). Linearity of expectation *always* holds, which makes it an extremely powerful tool.

6 Swaps and Cycles

Note 16

A permutation of n objects is a bijection from $(1,\ldots,n)$ to itself. For example, the permutation $\pi=(2,1,4,3)$ of 4 objects is the mapping $\pi(1)=2$, $\pi(2)=1$, $\pi(3)=4$, and $\pi(4)=3$. We'll say that a permutation $\pi=(\pi(1),\ldots,\pi(n))$ contains a *swap* if there exist $i,j\in\{1,\ldots,n\}$ so that $\pi(i)=j$ and $\pi(j)=i$, where $i\neq j$. The example above contains two swaps: $(1,2)$ and $(3,4)$.

(a) In terms of n, what is the expected number of swaps in a random permutation?

(b) In the same spirit as above, π contains a *k-cycle* if there exist $i_1,\ldots,i_k\in\{1,\ldots,n\}$ with $\pi(i_1)=i_2,\pi(i_2)=i_3,\ldots,\pi(i_k)=i_1$. What is the expected number of k-cycles?

Solution:

(a) As a warm-up, let's compute the probability that 1 and 2 are swapped. There are $n!$ possible permutations, and $(n-2)!$ of them have $\pi(1)=2$ and $\pi(2)=1$. This means

$$\mathbb{P}[(1,2)\text{ are a swap}]=\frac{(n-2)!}{n!}=\frac{1}{n(n-1)}.$$

There was nothing special about 1 and 2 in this calculation, so for any $\{i,j\} \subset \{1,\dots,n\}$, the probability that i and j are swapped is the same as above. Let's write $I_{i,j}$ for the indicator that i and j are swapped, and N for the total number of swaps, so that

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{\{i,j\}\subset\{1,\dots,n\}} I_{i,j}\right] = \sum_{\{i,j\}\subset\{1,\dots n\}} \mathbb{P}[(i,j) \text{ are swapped}] = \frac{1}{n(n-1)}\binom{n}{2} = \frac{1}{2}.$$

(b) The idea here is quite similar to the above, so we'll be a little less verbose in the exposition. However, as a first aside we need the notion of a *cyclic ordering* of k elements from a set $\{1,\dots n\}$. We mean by this a labelling of the k beads of a necklace with elements of the set, where we say that labellings of the beads are the same if we can move them along the string to turn one into the other. For example, $(1,2,3,4)$ and $(1,2,4,3)$ are different cyclic orderings, but $(1,2,3,4)$ and $(2,3,4,1)$ are the same. There are

$$\binom{n}{k}\frac{k!}{k} = \frac{n!}{(n-k)!}\frac{1}{k}$$

possible cyclic orderings of length k from a set with n elements, since if we first count all subsets of size k, and then all permutations of each of those subsets, we have overcounted by a factor of k.

Now, let N be a random variable counting the number of k-cycles, and for each cyclic ordering $(i_1,\dots,i_k)$ of k elements of $\{1,\dots,n\}$, let $I_{(i_1,\dots,i_k)}$ be the indicator that $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_k) = i_1$. There are $(n-k)!$ permutations in which $(i_1,\dots,i_k)$ form an k-cycle (since we are free to do whatever we want to the remaining $(n-k)$ elements of $\{1,\dots,n\}$), so the probability that $(i_1,\dots,i_k)$ are such a cycle is $\frac{(n-k)!}{n!}$, and

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{(i_1,\dots,i_k)\text{ cyclic ordering}} I_{(i_1,\dots,i_k)}\right] = \frac{n!}{(n-k)!}\frac{1}{k}\frac{(n-k)!}{n!} = \frac{1}{k}.$$