CS 70 离散数学 和 概率论
DIS 1A
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归纳 Intro

Natural numbers start at 0, 和 存在 always a next one. 对于 predicates on natural numbers the
Note3
principleofinductionis: n∈N,P(n)≡P(0)∧ n,P(n) =   P(n+1).
即, to 证明 P(n) 对于 natural numbers one proves P(0), the 基础情况, 和 n,P(n) =
P(n+1), the 归纳 step. In the 归纳 step, the assumption that P(n) is 真 is called the

inductionhypothesiswhichistypicallyusedtoarguethatP(n+1)istrue.
An 例子 is the statement P(n)=Σ n i= n(n+1) . The 基础情况, P(0), is the observation that
i=0 2
Σ 0 i=0. Intheinductionstep,theinductionhypothesis,P(n),isΣ
n i= (n)(n+1) . Theinduction
i=0 i=0 2
stepproceedsasfollows:
n+1 n (n)(n+1) (n+1)(n+2)
Σ i= Σi+n+1= +n+1= .
2 2
i=0 i=0

Thefirstequalityfollowsfromthedefinitionofthenotation,Σ,thesecondsubstitutestheinduction hypothesisandthelastisalgebra.
AndwhatisprovenisP(n+1),whichisthatΣ n+1i= (n+1)(n+2) .
i=0 2
Another 和 等价 view of the natural numbers are that 存在 the numbers 0 to n 和 那么
thereisn+1. Thestronginductionprincipleisthat
 n∈N,P(n)≡P(0)∧ n,((k≤n)P(k)) =   P(n+1).
Here the 归纳 假设 is that P(k) is 真 对所有 values k ≤n. To 证明 that every natural
number n≥2 can be written as a product of primes, we take the 基础情况 as P(2) 其中 can be
written as 2, 其中 is a product of a prime. 和 对于 any n, 如果 it is prime, it can be written as itself,
otherwise n=ab 和 by the inductive hypotheses P(a) 和 P(b) is that each can be written as a
product of primes. 因此, we can write n as the product of the primes in both a 和 b. 笔记 here
that the 基础情况 starts at 2, 其中 illustrates that one chose the 基础情况 as is relevant to the
statementbeingproven.

Strengtheningtheinductionhypothesisisatechniquethatprovesastrongertheorem. Forexample,
the notes 考虑 the 定理 "The sum of the first n odd numbers is a perfect square." In fact,
the notes inductively 证明 the stronger 定理 "The sum of the first n odd numbers is n2." Here,
the stronger inductive 假设 allows the 归纳 step to proceed easily. 笔记 that in strong
归纳, we 假设 more cases are 真 in the inductive 假设, whereas strengthening the

inductivehypothesisprovesastrongerclaimentirely.

1 Natural 归纳 on Inequality
Provethatifn∈Nandx>0,那么(1+x)n ⩾1+nx.
Note3
解答:
• BaseCase: Whenn=0,theclaimholdssince(1+x)0 ⩾1+0x.
• InductiveHypothesis: Assumethat(1+x)k ⩾1+kx
forsomevalueofn=k wherek∈N.
• InductiveStep: Forn=k+1,wecanshowthefollowing:
(1+x)k+1 =(1+x)k(1+x)⩾(1+kx)(1+x)
⩾1+kx+x+kx2
⩾1+(k+1)x+kx2 ⩾1+(k+1)x
Byinduction,wehaveshownthat n∈N,(1+x)n ⩾1+nx.
2 Make It Stronger
假设 that the 序列 a ,a ,... is defined by a =1 和 a =3a2 对于 n⩾1.
We want to
Note3 1 2 1 n+1 n
provethat
a
⩽3(2n)
n
foreverypositiveintegern.
(a) 假设 that we want to 证明 this statement using 归纳. Can we 令
our inductive
假设 be simply a ⩽
3(2n)?
Attempt an 归纳 证明 with this 假设 to 证明
n
whythisdoesnotwork.
(b)
Instead,令’strytoinsteadproveastrongerstatementinstead,oftheforma
⩽3(2n ??).
What
n
should replace the 问题 marks 所以 that the 归纳步骤 works out? Try
some examples
andseewhatworks.
(c)
Usingyourstrongerinductivehypothesisfrompart(b),carryouttheinductionproof.
(d) Whydoesthehypothesisinpart(b)implytheoverallclaim?
解答:
(a) 令’strytoprovethatforeveryn⩾1,wehavea
⩽32n
byinduction.
n
BaseCase: Forn=1wehavea
=1⩽321
=9.
1
InductiveStep: Forsomen⩾1,weassumea
⩽32n
. Now,considern+1. Wecanwrite:
n
a =3a2 ⩽3(32n)2 =3×32×2n =3×32n+1 =32n+1+1.
n+1 n
然而, what we wanted was to get an inequality of the form: a ⩽
32n+1
. 存在 an

n+1
extra+1intheexponentofwhatwederived.

(b) The 假设 should be a ≤
32n 1.
We can see in the next part that this modification
n
allowsthealgebraintheinductivesteptoworkoutnicely.
(c) Thistimetheinductionworks.
BaseCase: Forn=1wehavea =1≤32 1 =3.
1
InductiveStep: Forsomen≥1weassumea
≤32n 1.
Now,considern+1. Wecanwrite:
n
a =3a2 ≤3×(32n 1)2 =3×32×(2n 1) =3×32n+1 2 =32n+1 1.
n+1 n
Thisisexactlytheinductionhypothesisforn+1.
(d)
Foreveryn≥1,wehave2n 1≤2nandtherefore32n 1≤32n
. Thismeansthatourmodified
假设 其中 we proved in part (b) does indeed imply what we wanted
to 证明 in part
(a).
3 Binary Numbers
证明 that every 正的 整数 n can be written in binary. 换句话说, 证明
that 对于 any
Note3
positiveintegern,wecanwrite
n=c •2k+c •2k 1+• • • +c •21+c •20,
k k 1 1 0
forsomek∈Nandc ∈{0,1}foralli≤k.
i
As an 例子, the number 13 can be written in binary as 1101
because 13 = 1•23+1•22+0•
21+1•20.
解答:
Provebystronginductiononn.
The key insight here is that 如果 n is divisible by 2, 那么 it
is easy to get a bit string representation
of (n+1) from that of n. 然而, 如果 n is 非 divisible by 2, 那么
(n+1) will be, 和 its binary
representationwillbemoreeasilyderivedfromthatof(n+1)/2.
Moreformally:
• BaseCase:n=1canbewrittenas1×20.
• InductiveStep: 假设 that the statement is 真 对所有 1 ≤ m ≤ n,
其中 n is arbitrary.
Now, we need to 考虑 n+1. 如果 n+1 is divisible by 2, 那么 we can
apply our inductive

hypothesisto(n+1)/2anduseitsrepresentationtoexpressn+1inthedesiredform.
(n+1)/2=c •2k+c •2k 1+• • • +c •21+c •20
k k 1 1 0
n+1=2•(n+1)/2=c •2k+1+c •2k+• • • +c •22+c •21+0•20.
k k 1 1 0
Otherwise, n must be divisible by 2 和 因此 have c =0. We can
obtain the representation
0
ofn+1fromnasfollows:

n=c •2k+c •2k 1+•••+c •21+0•20
k k 1 1
n+1=c •2k+c •2k 1+•••+c •21+1•20
k k 1 1
因此,thestatementistrue.

Here is another alternate 解答 emulating the 算法 of converting
a decimal number to a
binarynumber.
• BaseCase:n=1canbewrittenas1×20.
•
InductiveStep:Assumethatthestatementistrueforall1≤m≤n,forarbitraryn.
Weshow
thatthestatementholdsforn+1.
Let2mbethelargestpowerof2suchthatn+1≥2m. 因此,
n+1<2m+1. Weexaminethenumber(n+1) 2m.
因为(n+1) 2m<n+1,theinductive
假设 holds, 所以 我们有 a binary representation 对于 (n+1) 2m. (如果
(n+1) 2m =0,
thenwestillhaveabinaryrepresentation,namely0•20.)
Also, 因为 n+1<2m+1, (n+1) 2m <2m, 所以 the largest power of 2
in the representation
of(n+1) 2m is2m 1. 因此,bytheinductivehypothesis,
(n+1) 2m =c •2m 1+c •2m 2+•••+c •21+c •20,
m 1 m 2 1 0
andadding2m tobothsidesgives
n+1=2m+c •2m 1+c •2m 2+•••+c •21+c •20,
m 1 m 2 1 0
whichisabinaryrepresentationforn+1.
因此,theinductioniscomplete.

Anotherintuitionisthatifxhasabinaryrepresentation,2xand2x+1doaswell:
shiftthebitsand
possibly place 1 in the last bit. The above 归纳 could 那么 have
proceeded from n 和 used
the binary representationof n/2 , shifting 和
possiblysetting the first bit depending on whether
nisoddoreven.
笔记: In proofs using simple 归纳, we only use P(n) in order to
证明 P(n+1). Simple
归纳 gets stuck here because in order to 证明 P(n+1) in the
归纳步骤, we need to
assumemorethanjustP(n).
Thisisbecauseitisnotimmediatelyclearhowtogetarepresentation
对于 P(n+1) using just P(n), particularly in the case that n+1
is divisible by 2. As a result, we

assumethestatementtobetrueforallof1,2,...,ninordertoproveitforP(n+1).
4 Fibonacci 对于 Home
Recall,theFibonaccinumbers,definedrecursivelyas
Note3
F =1,F =1,andF =F +F .
1 2 n n 2 n 1
ProvethateverythirdFibonaccinumberiseven. Forexample,F
=2isevenandF =8iseven.
3 6
解答:
Wewanttoprovethatforallnaturalnumbersk≥1,F iseven.
3k

Basecase: Fork=1,wecanseethatF =2iseven.
3
Inductionhypothesis:
Supposethatforanarbitraryfixedvalueofk,F iseven.
3k
Inductivestep: Wecanwrite
F =F +F =2F +F .
3k+3 3k+2 3k+1 3k+1 3k

Bytheinductionhypothesis,weknowthatF =2qforsomeq.
3k
This means that 我们有 that F = 2(F +q), 其中 蕴含 that it is even.
因此, by the
3k+3 3k+1
principlesofinductionwehaveshownthatallF areeven.
3k