CS 70 Discrete Mathematics and Probability Theory
Spring 2026 Sinclair, Song
DIS 8B

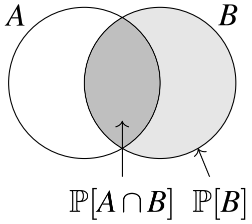
Conditional Probability Intro

Note 14

Conditional Probability: Probability of event A, *given* that event B has happened (implying that $P[B] > 0$);

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

Think of like restricting our sample space:



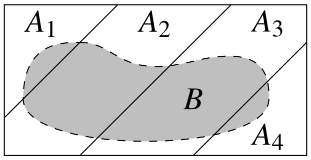
Bayes Rule: A consequence of conditional probability - notice $\mathbb{P}[A \cap B] = \mathbb{P}[A \mid B]\,\mathbb{P}[B] = \mathbb{P}[B \mid A]\,\mathbb{P}[A]$, so

$$\mathbb{P}[B \mid A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \frac{\mathbb{P}[A \mid B]\,\mathbb{P}[B]}{\mathbb{P}[A]}.$$

Total Probability Rule: If disjoint events $A_1, \ldots, A_n$ form a partition on the sample space Ω (meaning that each outcome in Ω belongs to exactly one of $A_1, \ldots, A_n$.), we then have

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \cap A_i] = \sum_{i=1}^{n} \mathbb{P}[B \mid A_i]\,\mathbb{P}[A_i].$$

Visually, we're splitting an event into partitions and looking at each intersection individually:



1 Box of Marbles

Note 14

You are given two boxes: one of them containing 900 red marbles and 100 blue marbles, the other one contains 500 red marbles and 500 blue marbles.

(a) If we pick one of the boxes randomly and pick a marble, what is the probability that it is blue?

(b) If we see that the marble is blue, what is the probability that it is chosen from box 1?

(c) Suppose we pick one marble from box 1 and without looking at its color we put it aside. Then we pick another marble from box 1. What is the probability that the second marble is blue?

Solution:

(a) Let B be the event that the picked marble is blue, R be the event that it is red, A_1 be the event that the marble is picked from box 1, and A_2 be the event that the marble is picked from box 2. Therefore we want to calculate $\mathbb{P}[B]$. By total probability,

$$\mathbb{P}[B] = \mathbb{P}[B \mid A_1]P[A_1] + \mathbb{P}[B \mid A_2]\,\mathbb{P}[A_2] = 0.5 \times 0.1 + 0.5 \times 0.5 = 0.3.$$

(b) In this part, we want to find $\mathbb{P}[A_1 \mid B]$. By Bayes rule,

$$\mathbb{P}[A_1 \mid B] = \frac{\mathbb{P}[B \mid A_1]\,\mathbb{P}[A_1]}{\mathbb{P}[B \mid A_1]\,\mathbb{P}[A_1] + \mathbb{P}[B \mid A_2]\,\mathbb{P}[A_2]} = \frac{0.1 \times 0.5}{0.5 \times 0.1 + 0.5 \times 0.5} = \frac{1}{6}.$$

(c) Let B_1 be the event that first marble is blue, R_1 be the event that the first marble is red, and B_2 be the event that second marble is blue without looking at the color of first marble. We want to find $\mathbb{P}[B_2]$. By total probability,

$$\mathbb{P}[B_2] = \mathbb{P}[B_2 \mid B_1]\,\mathbb{P}[B_1] + \mathbb{P}[B_2 \mid R_1]\,\mathbb{P}[R_1] = \frac{99}{999} \times 0.1 + \frac{100}{999} \times 0.9 = 0.1.$$

More generally, one can see that the probability that the n-th marble picked from box 1 is blue with probability 0.1. This is clear by symmetry: all the permutations of the 1000 marbles have the same probability, so the probability that the n-th marble is blue is the same as the probability that the first marble is blue.

2 Duelling Meteorologists

Note 14

Tom is a meteorologist in New York. On days when it snows, Tom correctly predicts the snow 70% of the time. When it doesn't snow, he correctly predicts no snow 95% of the time. In New York, it snows on 10% of all days.

(a) If Tom says that it is going to snow, what is the probability it will actually snow?

(b) Let A be the event that, on a given day, Tom predicts the weather correctly. What is $\mathbb{P}[A]$?

(c) Tom's friend Jerry is a meteorologist in Alaska. Jerry claims that she is a better meteorologist than Tom even though her overall accuracy is lower. After looking at their records, you determine that Jerry is indeed better than Tom at predicting snow on snowy days and sun on sunny day. Give an instance of the situation described above. This situation is actually an example of the famous Simpson's paradox!
Hint 1: "accuracy" refers to what we calculated in the previous part; it is the probability that

a weatherman predicts snow when it's snowy and sun when it's sunny. Try thinking about these two cases.

Hint 2: what is the weather like in Alaska, as compared to in New York?

Solution:

(a) Let S be the event that it snows and T be the event that Tom predicts snow.

$$\begin{aligned}\mathbb{P}[S|T] &= \frac{\mathbb{P}[S\cap T]}{\mathbb{P}[T]} \\ &= \frac{\mathbb{P}[S]\cdot\mathbb{P}[T|S]}{\mathbb{P}[S\cap T]+\mathbb{P}[\overline{S}\cap T]} \\ &= \frac{\frac{1}{10}\times\frac{7}{10}}{\frac{1}{10}\times\frac{7}{10}+\frac{9}{10}\times\frac{5}{100}} = \frac{14}{23}\end{aligned}$$

(b)

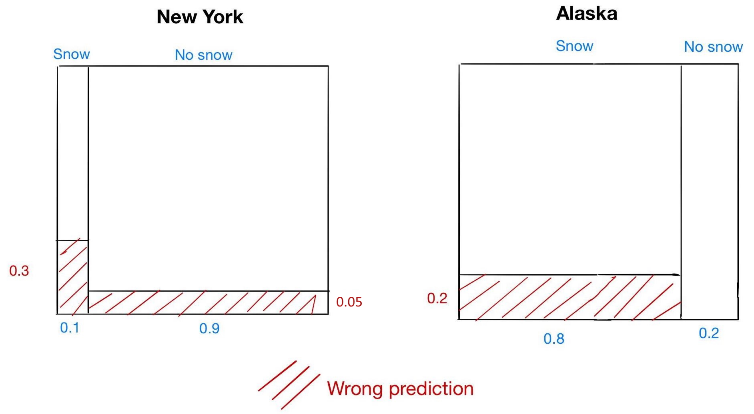
$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}[S\cap T]+\mathbb{P}[\overline{S}\cap\overline{T}] \\ &= \frac{1}{10}\times\frac{7}{10}+\frac{9}{10}\times\frac{95}{100} = \frac{37}{40}\end{aligned}$$

(c) Even though Jerry's overall accuracy is lower, it is still possible that she is a better meteorologist if the weather is different.

For example, let's assume that it snows 80% of days in Alaska.

- When it snows, Jerry correctly predicts snow 80% of the time.
- When it doesn't snow, Jerry correctly predicts no snow 100% of the time.

Jerry's overall accuracy turns out to be less than Tom's even though she is better at predicting both categories! The following diagram gives an illustration of the situation. The intuition is that Jerry's error gets penalized more heavily than Tom because it snows more often in Alaska.



For more info on this kind of phenomena, check out Simpson's Paradox!

3 Poisoned Smarties

Note 14

Supposed there are 3 people who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the owners, produces considerably more Smarties than her competitors and has a commanding 50% of the market share. Yousef See, who inherited her riches, lags behind Burr and produces 40% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 10%. However, a recent string of Smarties related food poisoning has forced the FDA to investigate these factories to find the root of the problem. Through her investigations, the inspector found that 2 Smarties out of every 100 at Kelly's factory was poisonous. At See's factory, 5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.1.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

(c) If a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

Solution:

(a) Let S be the event that a smarty is safe to eat. Let BK be the event that a smarty is from Burr Kelly's factory. Let YS be the event that a smarty is from Yousef See's factory. Finally, let SF be the event that a smarty is from Stan Furd's factory.

By total probability, we have

$$\begin{aligned}\mathbb{P}[S] &= \mathbb{P}[BK]\,\mathbb{P}[S \mid BK] + \mathbb{P}[YS]\,\mathbb{P}[S \mid YS] + \mathbb{P}[SF]\,\mathbb{P}[S \mid SF] \\ &= \frac{1}{2}\cdot\frac{49}{50} + \frac{2}{5}\cdot\frac{19}{20} + \frac{1}{10}\cdot\frac{9}{10} \\ &= \frac{49}{100} + \frac{38}{100} + \frac{9}{100} \\ &= \frac{96}{100} = \frac{24}{25} = 0.96\end{aligned}$$

Therefore the probability that a Smarty is safe to eat is 0.96.

(b) Let P be the event that a smarty is poisonous.

$$\mathbb{P}[P \mid \overline{BK}] = \frac{\mathbb{P}[\overline{BK} \cap P]}{\mathbb{P}[\overline{BK}]}$$

Since BK, YS, SF are a partition of the entire sample space, we know that if BK did not occur, then either YS occurred, or SF occurred:

$$\begin{aligned}
&= \frac{\mathbb{P}[YS \cap P]}{\mathbb{P}[\overline{BK}]} + \frac{\mathbb{P}[SF \cap P]}{\mathbb{P}[\overline{BK}]} \\
&= \frac{\mathbb{P}[P \mid YS]\,\mathbb{P}[YS]}{1 - \mathbb{P}[BK]} + \frac{\mathbb{P}[P \mid SF]\,\mathbb{P}[SF]}{1 - \mathbb{P}[BK]} \\
&= \frac{\frac{1}{20} \cdot \frac{2}{5}}{\frac{1}{2}} + \frac{\frac{1}{10} \cdot \frac{1}{10}}{\frac{1}{2}} = 2 \cdot \frac{2}{100} + 2 \cdot \frac{1}{100} \\
&= \frac{6}{100} = \frac{3}{50} = 0.06
\end{aligned}$$

(c) From Bayes' Rule, we know that:

$$\mathbb{P}[SF \mid P] = \frac{\mathbb{P}[P \mid SF]\,\mathbb{P}[SF]}{\mathbb{P}[P]}.$$

In part (a), we calculated the probability that any random Smarty was safe to eat; here, notice that $\mathbb{P}[P] = 1 - \mathbb{P}[S]$. This means we have

$$\begin{aligned}
\mathbb{P}[SF \mid P] &= \frac{\mathbb{P}[P \mid SF]\,\mathbb{P}[SF]}{1 - \mathbb{P}[S]} \\
&= \frac{\frac{1}{10} \cdot \frac{1}{10}}{1 - \frac{24}{25}} = \frac{\frac{1}{100}}{\frac{1}{25}} \\
&= \frac{25}{100} = \frac{1}{4} = 0.25
\end{aligned}$$